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Auxiliary field loop expansion of the effective action for a class of stochastic partial differential equations

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ABSTRACT

We present an alternative to the perturbative (in coupling constant) diagrammatic approach for studying stochastic dynamics of a class of reaction diffusion systems. Our approach is based on an auxiliary field loop expansion for the path integral representation for the generating functional of the noise induced correlation functions of the fields describing these systems. The systems we consider include Langevin systems describable by the set of self interacting classical fields $\phi_i(x, t)$ in the presence of external noise $\eta_i(x, t)$, namely $(\partial_t - \nu\nabla^2)\phi - F[\phi] = \eta$, as well as chemical reaction annihilation processes obtained by applying the many-body approach of Doi–Peliti to the Master Equation formulation of these problems. We consider two different effective actions, one based on the Onsager–Machlup (OM) approach, and the other due to Janssen–deGenneris based on the Martin–Siggia–Rose (MSR) response function approach. For the simple models we consider, we determine an analytic expression for the Energy landscape (effective potential) in both formalisms and show how to obtain the more physical effective potential of the Onsager–Machlup approach from the MSR effective potential in leading order in the auxiliary field loop expansion. For the KPZ equation we find that our approximation, which is non-perturbative and obeys broken symmetry Ward identities, does not lead to the appearance of

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a fluctuation induced symmetry breakdown. This contradicts the results of earlier studies.

1. Introduction

The effective action $\Gamma[\phi]$ for stochastic partial differential equations extends the role played by the action for field theories of classical dynamical systems. It is defined in terms of functional Legendre transformation of the generating functional for the connected correlation functions (see for example the book by Rivers [1]). The effective action accounts for the physics of the entire system, composed of the various degrees of freedom, represented by the fields, as well as the effect on them by stochastic agents in the form of noise. The first derivative of $\Gamma[\phi]$ with respect to the field gives the equation for the evolution of the field, averaged over different realizations of the noise chosen from a given probability distribution. Higher derivatives of the effective action determine the one particle-irreducible (1-PI) vertices, from which all the noise induced correlation functions of the field can be reconstructed. The 1-PI vertices play a crucial role in identifying the eventual renormalization of the parameters found in the theory without noise. The effective action in quantum field theory is reviewed in great detail in the books by Rivers [1] and Itzykson and Zuber [2], and the extension of this approach to studying reaction diffusion equations was pioneered by Hochberg and collaborators [3, 4]. For studying phase structure, one considers the energy landscape probed by homogeneous fields using the value of the effective action for homogeneous fields divided by the space time volume. The utility of the auxiliary field loop expansion is that in lowest order it leads to a self consistent Hartree-like approximation, which preserves underlying symmetries that give a qualitative analytic picture of the Energy Landscape. As a recent example, we have used it to give a simple qualitative picture of the phase structure of the Bose–Hubbard model [5]. When the dynamics leads to quartic (and higher) self interactions in the Lagrangian, auxiliary fields can convert the topology of the coupled field equations to be trilinear. This simplification leads to a dramatic topological simplification of the Schwinger–Dyson equations for the correlation functions. This also simplifies dramatically the expansion of the generating functional of the two particle irreducible graphs and allows one to study dynamics in approximations which in lowest order are related to the approach of Kraichnan [6,7] in his study of plasma turbulence. Related methods have recently been used by Doherty in the study of the Kardar–Parisi–Zhang (KPZ) equation [8].

Most text books on quantum field theory (see for example Ref. [2]) discuss the effective action and the effective potential in the semi-classical approximation, which keeps only the Gaussian fluctuations around the classical motion. Some more recent textbooks [9] describe approximations based on the generating functional for the two particle irreducible (2-PI) graphs, which includes the Hartree–Fock approximation. However approximations to this 2-PI generating functional are notorious for violating Ward–Takahashi identities [10]. This has been a great stumbling block for using the 2-PI approach when there are broken symmetries.

In the recent literature, most discussions of the effective action for stochastic partial differential equations are based on a loop expansion in terms of the strength of the noise correlation function. This approach is related to the semi-classical approximation to the effective action of quantum field theory. Another approach has been to use the self-consistent Hartree approximation. For the Kardar–Parisi–Zhang (KPZ) equation [11], recent discussions on the loop expansion are found in Ref. [3] and for the Hartree approximation [12]. In population biology, a recent discussion is found in Ref. [13], and for pair annihilation and Gribov processes a recent discussion is found in Ref. [4]. In the semi-classical approximation it is tacitly assumed that the noise is a small perturbation on the classical dynamics and that therefore perturbation theory in the strength of the noise is valid. This separation is often not present in many dynamical systems acted upon by noise, whether the noise is internal or external. This shortcoming of the semi-classical approximation was seen even at