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Beyond the Shannon–Khinchin formulation: The composability axiom and the universal-group entropy

Piergiulio Tempesta *

Departamento de Física Teórica II (Métodos Matemáticos de la física), Facultad de Físicas, Universidad Complutense de Madrid, 28040 Madrid, Spain
Instituto de Ciencias Matemáticas, C/ Nicolás Cabrera, No 13–15, 28049 Madrid, Spain

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ABSTRACT

The notion of entropy is ubiquitous both in natural and social sciences. In the last two decades, a considerable effort has been devoted to the study of new entropic forms, which generalize the standard Boltzmann–Gibbs (BG) entropy and could be applicable in thermodynamics, quantum mechanics and information theory. In Khinchin (1957), by extending previous ideas of Shannon (1948) and Shannon and Weaver (1949), Khinchin proposed a characterization of the BG entropy, based on four requirements, nowadays known as the Shannon–Khinchin (SK) axioms.

The purpose of this paper is twofold. First, we show that there exists an *intrinsic group-theoretical structure* behind the notion of entropy. It comes from the requirement of composability of an entropy with respect to the union of two statistically independent systems, that we propose in an axiomatic formulation. Second, we show that there exists a simple universal family of trace-form entropies. This class contains many well known examples of entropies and infinitely many new ones, a priori multi-parametric. Due to its specific relation with Lazard's universal formal group of algebraic topology, the new general entropy introduced in this work will be called the universal-group entropy. A new example of multi-parametric entropy is explicitly constructed.

* Correspondence to: Departamento de Física Teórica II (Métodos Matemáticos de la física), Facultad de Físicas, Universidad Complutense de Madrid, 28040 Madrid, Spain.

E-mail addresses: p.tempesta@fis.ucm.es, piergiulio.tempesta@icmat.es.

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1. Introduction

Entropy is a fundamental notion, at the heart of modern science. In the second half of the twentieth century, its range of applicability has been extended from the traditional context of classical thermodynamics to new areas such social sciences, economics, biology, quantum information theory, linguistics, etc. More recently, the role of entropy in the theory of complex systems has been actively investigated. From one side, several studies were devoted to axiomatic formulations, aiming at clarifying the foundational aspects of the notion of entropy. From the other side, many researchers pursued the idea of generalizing the classical Boltzmann–Gibbs statistical mechanics. Consequently, a plethora of new entropic forms, designed for extending the applicability of BG entropy to new contexts, was introduced.

The first research line was started by the seminal works by Shannon [1,2] and Khinchin [3]. A set of axioms, nowadays called the SK axioms, characterizing uniquely the BG entropy, was introduced (we shall make reference to the formulation of the axioms reported in Appendix A). The axioms (SK1)–(SK3) represent natural requirements (continuity, maximum principle, independence from zero probability events), that should be satisfied by any functional playing the role of an entropy. Instead, the axiom (SK4) simply characterizes the behavior of an entropy with respect to the composition of two systems, which reduces to additivity in the case of statistical independence of the systems.

For long time, additivity was interpreted as the property that ensures *extensivity*, i.e. the linear dependence of entropy on the number of particles of a system. Extensivity is crucial for an entropy to be thermodynamically admissible. Surprisingly, the two concepts are completely independent: additivity does not imply, nor is implied by extensivity. In addition, no entropy, irrespectively of being additive or nonadditive, can be extensive in any dynamical regime. For instance, if $W(N)$ is the total number of states of a complex system as a function of the number of its particles N, it turns out that a (sufficient) condition for the BG entropy to be extensive over the uniform distribution is that $W(N) \sim k^N$, with $k \in \mathbb{R}_+$; however, if $W(N) \sim N^k$, it is not.

The second research line, i.e. the study of generalized entropies and thermostatistics, in which the additivity axiom is explicitly violated, has become an extremely active research area in the last three decades. Since the work [4], many new entropic functionals have been proposed in the literature (see e.g. [5–16]). From the point of view of statistical mechanics, they may have a role in weakly chaotic regimes, when the ergodicity hypothesis is violated and the correlation functions exhibit a non-exponential decay, typically a power-law one [17]. In particular, Tsallis entropy, which is nonadditive, is extensive for special values of the parameter q in regimes where the BG entropy is not [18].

Another source of nonadditive entropies is Information Theory. In this context, generalized entropies can provide different versions of Kullback–Leibler-type divergences [19], useful for constructing comparative tests of sets of data.

In the study of entanglement, generalized entropies arise as useful alternatives to the von Neumann entropy [9,20]. As shown in [21], the von Neumann entropy may not avoid the detection of fake entanglement. The need for generalized nonadditive entropies in order to design efficient criteria for separability has been advocated in [9]. Recently, the relevance of generalized entropies in processes with quantum memory has been recognized [22].

In the present analysis of the mathematical foundations of the concept of entropy, we wish to point out the centrality of the notion of *composability*. We shall say that an entropy is composable if the following requirements are satisfied. First, given two statistically independent systems A and B, the entropy of the composed system $A \cup B$ depends on the entropies of the two systems $S(A)$ and $S(B)$ *only* (apart possibly a set of parameters). This is the original formulation of the concept, as e.g. in [23]. In addition, we require further properties, as the symmetry of a given entropy in the composition of the systems A and B, the stability of total entropy if one of the two systems is in a state of zero entropy, and the associativity of the composition of three independent systems (see axioms (C1)–(C4) below). Within this framework, the composability property, in this new sense, is equivalent to the existence of a *group-theoretical structure underlying the notion of entropy*, which guarantees the physical plausibility of the composition process.