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Multichannel framework for singular quantum mechanics

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HIGHLIGHTS

- A multichannel framework is proposed for singular quantum mechanics and analogues.
- The framework unifies several established approaches for singular potentials.
- Singular points are treated as new scattering channels.
- Nonunitary asymptotic behavior is subsumed in a unitary multichannel S-matrix.
- Conformal quantum mechanics and the inverse quartic potential are highlighted.

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ABSTRACT

A *multichannel S-matrix framework* for singular quantum mechanics (SQM) subsumes the renormalization and self-adjoint extension methods and resolves its boundary-condition ambiguities. In addition to the standard channel accessible to a distant ("asymptotic") observer, one supplementary channel opens up at each coordinate singularity, where local outgoing and ingoing *singularity waves* coexist. The channels are linked by a fully unitary S-matrix, which governs all possible scenarios, including cases with an *apparent* nonunitary behavior as viewed from asymptotic distances.

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1. Introduction

The central construct in quantum scattering processes is the S-matrix. However, while the orthodox formulation of the *S-matrix* is applicable to regular quantum mechanics and quantum field theory, its generalization for *singular quantum mechanics* (SQM) remains an open problem. In effect, the S-matrix orthodoxy is called into question due to the breakdown of the regular boundary condition at the singularity; this can be described as the loss of discriminating value when two linearly independent solutions remain equally acceptable for attractive singularities [1–5]. In addition, there is the issue of the possible emergence of nonunitary solutions [6,7] and the singular nature of the bound-state spectrum [8–16]. These extreme departures from regular quantum mechanics are evident for strongly singular attractive potentials—see definitions in Refs. [1,2], and in our Section 2.

In this paper we circumvent the divergence and unitarity challenges posed by SQM through a novel *multichannel framework* that comprehensively subsumes the well-established *renormalization schemes* of Refs. [8–14] and the method of *self-adjoint extensions* [15,16]. The latter is an efficient technique to handle this breakdown and the ensuing indeterminacy of the solutions by properly defining the domain of the Hamiltonian to guarantee self-adjointness. Our framework does provide a practical implementation of this technique, but also extends its usefulness beyond self-adjointness. In effect, the generalized framework consistently includes physical realizations with effective *absorption* [6,7] or *emission* by the singular potential—including, for example, the absorption of particles by charged nanowires (with conformal quantum mechanics) [17,18], the scattering by polarizable molecules (with an inverse quartic potential) [6], and miscellaneous applications to black holes [19–21] and D-branes [22–25].

In the proposed framework, a singularity point is treated as the opening of a new channel in lieu of a boundary condition. A consequence of this constructive approach is the existence of associated local outgoing and ingoing waves near the singularity—the natural generalization of the solutions used for the inverse quartic potential in Ref. [6] and the inverse square potential in Ref. [7]. In our framework, the crucial guiding criterion is the acceptance, on an equal footing, of these *"singularity waves"* as basic building blocks. In its final form, two related quantities are considered:

1. A unitary *multichannel S-matrix* [**S**], which effects a separation of the singular behavior at a point (for example, $r \sim 0$) from the longer-range properties of the interaction.
2. An *effective or asymptotic S-matrix* S_{asymp}, which directly yields the scattering observables viewed by an asymptotic observer according to the standard procedures of regular quantum mechanics.

While S_{asymp} may fail to be unitary due to the boundary condition ambiguities, the S-matrix [**S**] is guaranteed to satisfy unitarity for the quantum system that includes a channel connected to the singularity. In other words, when the singularity point is redefined as external to the given system (i.e., removed from the observable physics), *unitarity* is automatically restored within the enlarged multichannel system. Specifically, the S-matrix S_{asymp} is obtained via a Möbius transformation [26] of a complex-valued singularity parameter Ω (which specifies an auxiliary "boundary condition"), with coefficients provided by the multichannel S-matrix [**S**].

We have organized our paper as follows. In Section 2 we define singular potentials and SQM, and thereby discuss the boundary condition at the singularity and derive the existence of local ingoing and outgoing waves. The multichannel framework and properties of the S-matrix are introduced in Section 3, leading to the definition of the *effective asymptotic S-matrix* S_{asymp} in Section 4. In Section 5 we derive the multichannel S-matrix [**S**] for conformal quantum mechanics (Section 5.1) and for the inverse quartic potential (Section 5.2); for the conformal case, we extensively consider additional features arising from its SO(2,1) symmetry. The paper concludes in Section 6 with a comparative discussion of the multichannel S-matrix versus earlier approaches to SQM and related physical realizations. In the appendices, we discuss transformation properties and we include two models that modify conformal quantum mechanics in the infrared, verifying the robustness of the framework. Additional analytic properties and technical subtleties of this versatile framework are left for a forthcoming follow-up paper.