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A discussion on the origin of quantum probabilities

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HIGHLIGHTS

- Several recent works use a derivation similar to that of R.T. Cox to obtain quantum probabilities.
- We apply Cox's method to the lattice of subspaces of the Hilbert space.
- We obtain a derivation of quantum probabilities which includes mixed states.
- The method presented in this work is susceptible to generalization.
- It includes quantum mechanics and classical mechanics as particular cases.

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ABSTRACT

We study the origin of quantum probabilities as arising from non-Boolean propositional-operational structures. We apply the method developed by Cox to non distributive lattices and develop an alternative formulation of non-Kolmogorovian probability measures for quantum mechanics. By generalizing the method presented in previous works, we outline a general framework for the deduction of probabilities in general propositional structures represented by lattices (including the non-distributive case).

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1. Introduction

Quantum probabilities[1] posed an intriguing question from the very beginning of quantum theory. It was rapidly realized that probability amplitudes of quantum process obeyed rules of a non classical nature, as for example, the sum rule of probability amplitudes giving rise to interference terms or the nonexistence of joint distributions for noncommuting observables. In 1936 von Neumann wrote the first work ever to introduce quantum logics [1–3], suggesting that quantum mechanics requires a propositional calculus substantially different from all classical logics. He rigorously isolated a new algebraic structure for quantum logics, and studied its connections with quantum probabilities. Quantum and classical probabilities have points in common as well as differences. These differences and the properties of quantum probabilities have been intensively studied in the literature [4–14]. It is important to remark that not all authors believe that quantum probabilities are essentially of a different nature than those which arise in probability theory (see for example [15] for a recent account). Though this is a major question for probability theory and physics, it is not our aim in this work to settle this discussion.

There exist two important axiomatizations of classical probabilities. One of them was provided by Kolmogorov [16], a set theoretical approach based on Boolean sigma algebras of a sample space. Probabilities are defined as measures over subsets of a given set. Thus, the Kolmogorovian approach is set theoretical and usually identified (but not necessarily) with a frequentistic interpretation of probabilities. Some time later it was realized that quantum probabilities can be formulated as measures over non Boolean structures (instead of Boolean sigma algebras). This is the origin of the name "non-Boolean or non-Kolmogorovian" probabilities [8]. It is remarkable that the creation of quantum theory and the works on the foundations of probability by Kolmogorov were both developed at the same time, in the twenties.

An alternative approach to the Kolmogorovian construction of probabilities was developed by R.T. Cox [17,18]. Cox starts with a propositional calculus, intended to represent assertions which portray our knowledge about the world or system under investigation. As it is well known since the work of Boole [19], propositions of classical logic (CL) can be represented as a Boolean lattice, i.e., an algebraic structure endowed with lattice operations "$\wedge$", "$\vee$", and "$\neg$", which are intended to represent conjunction, disjunction, and negation, respectively, together with a partial order relation "$\leq$" which is intended to represent logical implication. Boolean lattices (as seen from an algebraic point of view) can be characterized by axioms [20–22]. By considering probabilities as an inferential calculus on a Boolean lattice, Cox showed that the axioms of classical probability can be deduced as a consequence of lattice symmetries, using entropy as a measure of information. Thus, differently form the set theoretical approach of Kolmogorov, the approach by Cox considers probabilities as an inferential calculus.

It was recently shown that Feynman's rules of quantum mechanics can be deduced from operational lattice structures using a variant of Cox's method [23,24,15,25,26] (see also [20,21]). For example, in [15,25] this is done by:

- first defining an operational propositional calculus on a quantum system under study, and after that,
- postulating that any quantum process (interpreted as a proposition in the operational propositional calculus) can be represented by a pair of real numbers and,
- using a variant of the method developed by Cox, showing that these pairs of real numbers obey the sum and product rules of complex numbers, and can then be interpreted as the quantum probability amplitudes which appear in Feynman's rules.

There is a long tradition with regards to the application of lattice theory to physics and many other disciplines. The quantum logical (QL) approach to quantum theory (and physics in general),

[1] By the term "quantum probabilities", we mean the probabilities that appear in quantum theory. As is well known, they are ruled by the well known formula $\mathrm{tr}(\rho P)$, where ρ is a density matrix representing a general quantum state and P is a projection operator representing an event (see Section 4 of this work for details).