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On solvability and integrability of the Rabi model

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HIGHLIGHTS

- Schweber's criterion shown equivalent to a meromorphic function F with real simple poles and positive residues.
- Calculation of spectra determined as zeros of F greatly facilitated: one has exactly one zero between subsequent poles of F.
- Spectrum in a given parity eigenspace is necessarily nondegenerate.
- Recent claims regarding solvability and integrability of the Rabi model found to be largely unsubstantiated.

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ABSTRACT

The quasi-exactly solvable Rabi model is investigated within the framework of the Bargmann Hilbert space of analytic functions $\mathcal{B}$. On applying the theory of orthogonal polynomials, the eigenvalue equation and eigenfunctions are shown to be determined in terms of three systems of monic orthogonal polynomials. The formal Schweber quantization criterion for an energy variable x, originally expressed in terms of infinite continued fractions, can be recast in terms of a meromorphic function $F(z) = a_0 + \sum_{k=1}^{\infty} \mathcal{M}_k/(z - \xi_k)$ in the complex plane $\mathbb{C}$ with *real simple* poles ξ_k and *positive* residues $\mathcal{M}_k$. The zeros of $F(x)$ on the real axis determine the spectrum of the Rabi model. One obtains at once that, on the real axis, (i) $F(x)$ monotonically decreases from $+\infty$ to $-\infty$ between any two of its subsequent poles ξ_k and ξ_{k+1}, (ii) there is exactly one zero of $F(x)$ for $x \in (\xi_k, \xi_{k+1})$, and (iii) the spectrum corresponding to the zeros of $F(x)$ does not have any accumulation point. Additionally, one can provide a much simpler proof that the spectrum in each parity

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eigenspace $\mathcal{B}_\pm$ is necessarily *nondegenerate*. Thereby the calculation of spectra is greatly facilitated. Our results allow us to critically examine recent claims regarding solvability and integrability of the Rabi model.

1. Introduction

The Rabi model [1] describes the simplest interaction between a cavity mode with a bare frequency ω and a two-level system with a bare resonance frequency ω_0. The model is characterized by the Hamiltonian [1–4]

$$\hat{H}_R = \hbar\omega \mathbb{1}\hat{a}^\dagger\hat{a} + \hbar g\sigma_1(\hat{a}^\dagger + \hat{a}) + \mu\sigma_3, \tag{1}$$

where $\mathbb{1}$ is the unit matrix, $\hat{a}$ and $\hat{a}^\dagger$ are the conventional boson annihilation and creation operators satisfying commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, g is a coupling constant, and $\mu = \hbar\omega_0/2$. In what follows we assume the standard representation of the Pauli matrices σ_j and set the reduced Planck constant $\hbar = 1$. For dimensionless coupling strength $\kappa = g/\omega \lesssim 10^{-2}$, the physics of the Rabi model is well captured by the analytically solvable approximate Jaynes and Cummings (JC) model [5,6]. The latter is obtained from the former upon applying the rotating wave approximation (RWA), whereby the coupling term $\sigma_1(\hat{a}^\dagger + \hat{a})$ in Eq. (1) is replaced by $(\sigma_+\hat{a} + \sigma_-\hat{a}^\dagger)$, where $\sigma_\pm \equiv (\sigma_1 \pm i\sigma_2)/2$. Nowadays, solid-state semiconductor [7] and superconductor systems [8–10] have allowed the advent of the *ultrastrong* coupling regime, where the dimensionless coupling strength $\kappa \gtrsim 0.1$ [11]. In this regime, the validity of the RWA breaks down and the relevant physics can only be described by the full Rabi model [1]. With new experiments rapidly approaching the limit of the *deep strong* coupling regime characterized in that $\kappa \gtrsim 1$ [12], i.e., an order of magnitude stronger coupling, the relevance of the Rabi model [1] becomes even more prominent. There is every reason to believe that ultrastrong and deep strong coupling systems could open up a rich vein of research on truly quantum effects with implications for quantum information science and fundamental quantum optics [7].

The Rabi model applies to a great variety of physical systems, including cavity and circuit quantum electrodynamics, quantum dots, polaronic physics and trapped ions. In spite of recent claims [3,13], the model is *not* exactly solvable. Rather it is a typical example of *quasi-exactly solvable* (QES) models in quantum mechanics [14–18]. The QES models are distinguished by the fact that a *finite* number of their eigenvalues and corresponding eigenfunctions can be determined algebraically [14–17]. That is also the case of the Rabi model [18]. Certain energy levels of the Rabi model, known as Juddian exact isolated solutions [19], can be analytically computed [19–21], whereas the remaining part of the spectrum not [20,21]. Depending on model parameters, the spectrum can only be approximated (sometime rather accurately—cf. Eqs. (18), (20) and Fig. 3 of Ref. [22]; Eq. (20) and Figs. 1, 2 of Ref. [23]). Therefore, any kind of exact results involving the Rabi model continues to be of great theoretical and experimental value.

In our earlier work [24] we studied the Rabi model as a member of a more general class $\mathcal{R}$ of quantum models. In the Hilber space $\mathcal{B} = L^2(\mathbb{R}) \otimes \mathbb{C}^2$, where $L^2(\mathbb{R})$ is represented by the Bargmann space of entire functions $\mathfrak{b}$, and $\mathbb{C}^2$ stands for a spin space [2,25], the models of the class $\mathcal{R}$ were characterized in that the eigenvalue equation

$$\hat{H}\Phi = E\Phi, \tag{2}$$

where $\hat{H}$ denotes a corresponding Hamiltonian, reduces to a *three-term difference equation*

$$\phi_{n+1} + a_n\phi_n + b_n\phi_{n-1} = 0 \qquad (n \geq 0). \tag{3}$$

Here $\{\phi_n\}_{n=0}^\infty$ are the sought expansion coefficients of an entire function

$$\varphi(z) = \sum_{n=0}^\infty \phi_n z^n \tag{4}$$