Annals of Physics 327 (2012) 2050–2059

Contents lists available at SciVerse ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

An improved proximity force approximation for electrostatics

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ARTICLE INFO

Article history:
Received 16 January 2012
Accepted 22 April 2012
Available online 2 May 2012

Keywords:
Proximity forces
Electrostatics
Casimir effect

ABSTRACT

A quite straightforward approximation for the electrostatic interaction between two perfectly conducting surfaces suggests itself when the distance between them is much smaller than the characteristic lengths associated with their shapes. Indeed, in the so called "proximity force approximation" the electrostatic force is evaluated by first dividing each surface into a set of small flat patches, and then adding up the forces due two opposite pairs, the contributions of which are approximated as due to pairs of parallel planes. This approximation has been widely and successfully applied in different contexts, ranging from nuclear physics to Casimir effect calculations. We present here an improvement on this approximation, based on a derivative expansion for the electrostatic energy contained between the surfaces. The results obtained could be useful for discussing the geometric dependence of the electrostatic force, and also as a convenient benchmark for numerical analyses of the tip–sample electrostatic interaction in atomic force microscopes.

1. Introduction

A standard problem in electromagnetism is that of computing the electrostatic force between conducting bodies, or its close relative: the calculation of the capacitance of a system of conductors. The simplest example is the case of parallel plates separated by a distance much smaller than the

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doi:10.1016/j.aop.2012.04.006

characteristic size of the plates, which are held at a fixed electrostatic potential difference. Albeit not as straightforwardly as in that example, some other systems admit analytical exact solutions also; indeed, that is the case for two eccentric cylinders, for a cylinder in front of a plane, and also for a sphere in front of a plane.

This problem is also of considerable practical relevance in electrostatic force microscopy (EFM) and its variants [1], which are based on the interaction between biased atomic force microscopy (AFM) tips and a sample. The same applies to the experimental determination of Casimir or gravitational forces between conducting bodies, as residual charges or potentials produce unwanted forces that must be subtracted in order to determine the sought-after force [2].

Of course the electrostatic force between bodies of arbitrary shape can, in principle, be computed by solving numerically the Laplace equation with adequate boundary conditions. However, analytic or semi-analytic methods are always welcome, as ways to improve the understanding of the geometric dependence of the force, and also to be used as simple benchmarks of fully numerical computations. For instance, in the context of EFM/AFM, analytical models for the tips have been developed, and additional exactly solvable models have been found, like the case of a hyperboloid in front of a plane [3]. Generalized image-charge methods have also been proposed [4], in which the tip and the sample are replaced by a set of fictitious charges, whose intensity and positions are found numerically.

An interesting analytical approach was introduced by Derjaguin in 1934; Derjaguin developed an approximate method for computing the Van der Waals force between macroscopic bodies assumed to be close to each other [5,6]. The approximation assumes that the surfaces of the bodies can be replaced by a set of flat patches and that at short distances the dominant contributions correspond to pairs of patches (one on each surface) which are closest to each other. Moreover, the interaction is supposed to be additive. In this way, it is possible to compute the force between gently curved surfaces from the knowledge of the interaction energy for flat surfaces, as long as the radii of curvature of the surfaces are much larger than the minimum distance between bodies and when the surface normals in opposite patches are almost parallel. Later on, the same idea was applied in nuclear physics, under the name of the proximity force approximation (PFA) or proximity force theorem, in order to compute the interaction between nuclei [7,8]. It has also been widely used to compute Casimir (or retarded Van der Waals) forces between neutral macroscopic objects [2]. It should be clear that, *mutatis mutandis*, the Derjaguin approximation and its ulterior developments can also be applied in the analysis of electrostatic forces, at least between gently curved conducting surfaces.

In spite of the simplicity and usefulness of the PFA, for many years there was a stumbling block impeding its progress, since methods for assessing its reliability or computing the next to leading order (NTLO) correction were lacking. This situation left, as the only alternative for assessing the PFA reliability, its comparison with the rather few examples for which an exact result was available.

In an attempt to improve that situation, we have recently shown [9], in the context of Casimir physics, that the PFA can be considered as the first term in an expansion in derivatives of the surface shapes of the interaction energy. In this way, it is now possible to improve the PFA by computing the NTLO corrections.

In this paper we apply this idea in the context of electrostatics, as already suggested in [9]. Our aim is twofold: on the one hand, to show the potential usefulness of the improved PFA in computing electrostatic forces. On the other hand, we believe that this attempt to present the improved PFA in a simpler context may help one to gain intuition about its applicability in more complex scenarios.

2. The proximity force approximation

Let us assume, for the sake of simplicity, that the system consists of a gently varying surface in front of a plane (both perfect conductors). The plane is at $z = 0$, and we assume it to be grounded. The curved surface is described by a single function $z = \psi(x, y)$, and is assumed to be at a constant electrostatic potential V. We shall use the notation $x_\parallel = (x, y)$. The electrostatic energy contained between surfaces is then given by

$$U = \frac{\epsilon_0}{2} \int d^2x_\parallel \int_0^{\psi(x_\parallel)} dz\, |\mathbf{E}|^2. \tag{1}$$