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Topology, and (in)stability of non-Abelian monopoles

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ABSTRACT

The stability problem of non-Abelian monopoles with respect to "Brandt–Neri–Coleman type" variations reduces to that of a pure gauge theory on the two-sphere. Each topological sector admits exactly one stable monopole charge, and each unstable monopole admits $2\sum(2|q|-1)$ negative modes, where the sum goes over the negative eigenvalues q of an operator related to the non-Abelian charge $\mathbb{Q}$ of Goddard, Nuyts and Olive. An explicit construction for the [up-to-conjugation] unique stable charge, as well as the negative modes of the Hessian at any other charge is given. The relation to loops in the residual group is explained. From the global point of view, the instability is associated with energy-reducing two-spheres, which, consistently with the Morse theory, generate the homology of the configuration space. Our spheres are tangent to the negative modes at the considered critical point, and may indicate possible decay routes of an unstable monopole as a cascade into lower lying critical points.

Contents

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1. Introduction: stability

Magnetic monopoles arise as exact solutions of spontaneously broken Yang–Mills–Higgs theory [1–5], see Section 2 for an outline. It has been pointed out by Brandt and Neri [6] and emphasized by Coleman [3], however, that most such solutions are unstable when the residual gauge group H is non-Abelian.

This review, which heavily draws on previous work of two of us with late O'Raifeartaigh, [7,8], is devoted to the study of various aspects of "Brandt–Neri–Coleman" monopole instability. Further related contributions can be found in [9–11].

1.1. Local aspects: the Hessian

The intuitive picture behind the stability problem is that of the Morse theory [12]. The Yang–Mills–Higgs energy functional, $\mathcal{E}$, can be viewed as a "surface" above the (infinite dimensional) "manifold of static field configurations" $\mathcal{C}$. Static solutions (like monopoles) of the Yang–Mills–Higgs field equations are *critical points* of $\mathcal{E}$ i.e. points where the gradient of $\mathcal{E}$ vanishes,

$$\delta\mathcal{E} = 0. \tag{1.1}$$

These critical points can be local minima [or maxima], or saddle points and can also be degenerate, meaning that it belongs to a submanifold with constant value of $\mathcal{E}$. The theoretically possible "landscapes" are, hence, as depicted on Figs. 1 and 2.

The nature of the critical point can be tested by considering small oscillations around it: for a minimum, represented by the bottom of a "cup" (Fig. 1a), all oscillations would *increase* the energy. Such a configuration is classically *stable*.

For a saddle point (Fig. 1b) some oscillations would increase the energy; these are the stable modes. Some other ones would instead *decrease* the energy: there exist *negative modes*.

A critical point can also be degenerate, meaning that one may have *zero modes*, i.e. oscillations which leave the energy unchanged, cf. Fig. 2.

The intuitive picture is that if one puts a ball into a critical point, it will roll down along energy-reducing directions, — except when it is a (local) minimum and no such directions exist.

How can we determine the type of a critical point? In finite dimensions, we would use differential calculus: a critical point is where all first partial derivatives vanish. Then the behaviour of oscillations depends on the matrix of second derivatives called the *Hessian*,

$$\delta^2\mathcal{E} = \left[\partial_i\partial_j\mathcal{E}\right]. \tag{1.2}$$

$\delta^2\mathcal{E}$ defines a symmetric quadratic form which is positive or negative definite if it is a [local] minimum or maximum, indefinite for a saddle point and degenerate if it has energy-preserving deformations. All this can be detected by looking at the eigenvalues of $\delta^2\mathcal{E}$: are they all positive, or both positive or negative, or do we have zero eigenvalues.