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From entanglement renormalisation to the disentanglement of quantum double models

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ABSTRACT

We describe how the entanglement renormalisation approach to topological lattice systems leads to a general procedure for treating the whole spectrum of these models in which the Hamiltonian is gradually simplified along a parallel simplification of the connectivity of the lattice. We consider the case of Kitaev's quantum double models, both Abelian and non-Abelian, and we obtain a rederivation of the known map of the toric code to two Ising chains; we pay particular attention to the non-Abelian models and discuss their space of states on the torus. Ultimately, the construction is universal for such models and its essential feature, the lattice simplification, may point towards a renormalisation of the metric in continuum theories.

1. Introduction

The application of tensor network methods has provided deep insight into topologically ordered systems. Tensor networks provide a setting where the exotic characteristics of topological order (topology dependent ground level degeneracy, local indistinguishability of ground states, topological entanglement entropy, interplay with renormalisation, unusual realisation of symmetries) can be exactly studied. On the other hand, tensor network methods are sufficiently flexible for studying the excitations of these systems; in 2D, as is known, the excitations are quasiparticles with anyonic exchange statistics, which is the basis of Kitaev's topological quantum computer architecture [1].

Among lattice models with topological order, Kitaev's quantum double models [1] have a distinguished position. They were the first models in which the possibilities of the topological setting

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for quantum computation purposes were discussed, and include anyon models universal for quantum computation by braiding; on the other hand, quantum double models bear an intimate relation to discrete gauge theories, exhibiting a rich group-theoretical and algebraic structure that pervades the study of their ground level and excitations, and in particular determines the properties of the anyons and their computational power. While they are not as general as string-net models [2], the models that describe all doubled topological fixed points on the lattice, steps towards a reformulation of the latter in the spirit of quantum double models have been taken in [3–5]. Recent progress in the perturbative production of quantum double codes via two-body interactions has been reported in [6].

Tensor network methods were first applied to topological models on the lattice in [7], where an exact projected entangled-pair state (PEPS) representation of a ground state of the toric code was given. In recent years, a growing number of tensor network representations for topological states have been developed following the lead of [7] (e.g. [8–11,4]), which are notably exact for ground states of fixed-point Hamiltonians (quantum doubles, string-nets), and moreover lend themselves to deep analysis of their symmetries [12]. Tensor network methods for systems with anyonic quasiparticles, independent of any microscopic structure, have also been developed [13,14].

The multi-scale entanglement renormalisation ansatz (MERA) is a particular kind of tensor network structure, developed by Vidal [15,16], which builds on the coarse-graining procedure typical for real-space renormalisation flows, but introduces layers of unitary operators between coarse-graining steps so as to reduce interblock entanglement, a procedure called entanglement renormalisation (ER). In practice, ER led to tractable MERA representations and algorithms for ground states of critical systems in 1D, traditionally hard for tensor network methods, and the numerical applications of the method now encompass a wide class of lattice and condensed-matter models both in 1D and in higher dimensions. ER was first applied to topological lattice models in [8], where exact MERAs were given for ground states of quantum double models, both Abelian and non-Abelian. In [9] ground states of string-net models were also written in MERA form.

While understanding ground states of many-body quantum systems is certainly of the utmost importance, it is a bonus for a method to be able to explore at least the low-lying excitation spectrum. The purpose of this paper is to show how the principles of the ER approach to quantum double models can be extended to account for the whole spectrum in an exact way. The picture that emerges is somehow complementary to entanglement renormalisation: by giving up the coarse-graining, that is, using unitary tensors throughout, the ER method turns into a reorganisation of degrees of freedom which is essentially graph-theoretical and proceeds by modifying the lattice by keeping the topological Hamiltonian the same at each step; this procedure yields finally a Hamiltonian consisting essentially of one-body terms, or an essentially classical one-dimensional spin chain model.

Naturally, the structure of the tensor network with unitary operations defines a unitary quantum circuit, which in this case simplifies the structure of the Hamiltonian down to mostly one-body terms. In this sense, the construction can be regarded as an instance of the broad idea of quantum circuits diagonalising Hamiltonians, introduced in [17].

In this vein, we obtain an explicit (and geometrically appealing) construction of the well known mapping of the toric code to two classical Ising chains; but we also cover the non-Abelian cases. Remarkably, the construction is virtually universal for all quantum double models, and can be understood as a series of moves simplifying the structure of the lattice into a series of 'bubbles' and 'spikes', each one of which talks only to one qudit; each quantum double model possesses a canonical set of operations transforming the model on one lattice into the same model on the next lattice. Since the lattice structure can be considered as the discretisation of a metric, this might open the door to speculations about a continuum counterpart (including, perhaps, dimensional reduction in going from the topological Hamiltonian in 2D to one-dimensional classical chains). We remark that the recent paper [18] uses related ideas to find numerical methods applicable to lattice gauge theories.

The structure of the paper is as follows. The Abelian case of the toric code is discussed in Section 2, where we recall the basic elements of the construction of [8], and proceed to a detailed presentation of the disentangling method. In Section 3, the analogous construction is developed for general quantum double models; particular attention is given to the models on the torus, where the structure of the Hilbert space has to be taken into account carefully. Section 4 contains a discussion and conclusions.