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Weak chaos in the disordered nonlinear Schrödinger chain: Destruction of Anderson localization by Arnold diffusion

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ABSTRACT

The subject of this study is the long-time equilibration dynamics of a strongly disordered one-dimensional chain of coupled weakly anharmonic classical oscillators. It is shown that chaos in this system has a very particular spatial structure: it can be viewed as a dilute gas of chaotic spots. Each chaotic spot corresponds to a stochastic pump which drives the Arnold diffusion of the oscillators surrounding it, thus leading to their relaxation and thermalization. The most important mechanism of equilibration at long distances is provided by random migration of the chaotic spots along the chain, which bears analogy with variable-range hopping of electrons in strongly disordered solids. The corresponding macroscopic transport equations are obtained.

1. Introduction

Anderson localization [1] is a general phenomenon occurring in many linear wave-like systems subject to a disordered background (see Ref. [2] for a recent review). It is especially pronounced in one-dimensional systems, where even an arbitrarily weak disorder localizes all normal modes of the system [3,4]. Anderson localization in linear systems (single-particle problems in quantum mechanics) has been thoroughly studied over the last 50 years; its physical picture is quite clear by now (although some open questions still remain), and even rigorous mathematical results have been established. The situation is much less clear in the presence of nonlinearities/interactions.

One of the simplest systems where the effect of a classical nonlinearity on the Anderson localization can be studied, is the disordered nonlinear Schrödinger equation (DNLSE) in one dimension, discrete or continuous. It has attracted a lot of interest in the last few years, including both stationary and non-stationary problems, as well as the problem of the dynamic stability of stationary solutions [5–37].

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This interest was especially motivated by the recent experimental observation of the Anderson localization of light in disordered photonic lattices [38–40], where DNLSE describes light propagation in the paraxial approximation, and of cold atoms in disordered optical lattices [41–43], for which DNLSE provides the mean-field description. Other systems have also been studied, such as a chain of anharmonic oscillators [5,15,44,45,20,21,32,33,35,46], a chain of classical spins [5,47], a nonlinear Stark ladder [48,49].

In contrast to linear problems which can always be reduced to finding the normal modes and the corresponding eigenvalues, the problem of localization in a nonlinear system can be stated in different inequivalent ways. For example, one can study solutions of the stationary nonlinear Schrödinger equation with disorder [12,14,28]; however, in a nonlinear system they are not directly related to the dynamics. Another possible setting is the system subject to an external perturbation or probe; studies of the transmission of a finite-length sample where an external flow is imposed [6,8–11,19,25,27], or dipolar oscillations in an external trap potential [18], fall into this category. Most attention has been paid to the problem of spreading of an initially localized wave packet of a finite norm [7–9,13,15–17,50,20–23,48,49,24,29–34] (in linear systems it remains exponentially suppressed at long distances for all times). This setting corresponds directly to experiments [38–42]. A closely related but not equivalent problem is that of thermalization of an initially out-of-equilibrium system (in a linear localized system thermalization does not occur). The difference between these two settings is that in the latter the total energy stored in the system is proportional to its (infinite) size, while in the former the infinite system initially receives a finite amount of energy. The problem of thermalization has also received attention [44,45,47,26], and it is the main subject of the present work (although the problem of wave packet spreading will also be briefly discussed). Still, despite a large body of work, detailed understanding of the equilibration dynamics in one-dimensional disordered nonlinear systems is still lacking.

On the one hand, the direct numerical integration of the differential equation shows that an initially localized wave packet of a finite norm does spread indefinitely, its size growing with time as a sub-diffusive power law [7,9,15,16,20,21,32,33,35]. Existence of different regimes of spreading, depending on whether the system is in the regime of strong or weak chaos, has been discussed [33,35]. Several authors suggested a macroscopic description of the long-time dynamics in terms of a nonlinear diffusion-type equation resulting in sub-diffusive spreading of the wave packet [33,31,34]; however, arguments used to justify this equations are based on an *a priori* assumption of spatially uniform chaos.

On the other hand, it was argued that among different initial conditions with a finite norm, at least some should exhibit regular quasi-periodic dynamics, and thus would not spread indefinitely [5,13,37]. For finite-size systems, scaling of the probability for the system to be in the chaotic or regular regime has been studied numerically [36]. When scaling arguments are applied to the results of the numerical integration of the equation of motion, they indicate slowing down of the power-law spreading [34]. Rigorous mathematical arguments support the conclusion that at long times the wave packet should spread (if spread at all) slower than any power law [50].

The aim of the present work is the analysis of statistical properties of weak chaos in the discrete one-dimensional disordered nonlinear Schrödinger equation with the initial conditions corresponding to finite norm and energy *densities*. The main focus is on long-time dynamics and relaxation at large distances, when no stationary superfluid flow can exist in the system [11,27] and the dynamics is chaotic. It is often assumed that upon thermalization chaos has no spatial structure, and all sites of the chain are more or less equally chaotic; here it is argued not to be the case.[1] Namely, in the regime of strong disorder and weak nonlinearity chaos is concentrated on a small number of rare chaotic spots (essentially the same picture was also proposed in Ref. [47] for a disordered chain of coupled classical spins, but it was not put on a quantitative basis). A chaotic spot is a collection of resonantly coupled oscillators, in which one can separate a collective degree of freedom (namely, their relative phase whose dynamics is slow because of the resonance), which performs chaotic motion. Under the conditions of weak coupling between neighboring oscillators and weak nonlinearity (i.e., Anderson localization being

[1] We will call the motion of an oscillator the more chaotic the faster it loses the phase memory.