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A dispersion relation for the density of states with application to the Casimir effect

S.G. Rajeev

Department of Physics and Astronomy, Department of Mathematics, University of Rochester, Rochester, NY 14627, USA

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ABSTRACT

The trace of a function of a Schrödinger operator minus the same for the Laplacian can be expressed in terms of the determinant of its scattering matrix. The naive formula for this determinant is divergent. Using a dispersion relation, we find another expression for it which is convergent, but needs one piece of information beyond the scattering matrix: the spatial integral of the potential. Except for this 'anomaly', we can express the Casimir energy of a compact body in terms of its optical scattering matrix, without assuming any rotational symmetry for its shape.

1. Introduction

The kinetic theory of gases in classical physics predicts that every surface in a gas is bombarded by molecules. The recoil of these molecules exert a pressure on the surface. A spectacular demonstration of this prediction would be to evacuate the air inside an aluminium can: there will be a net inward pressure that will cause the can to collapse.

In the quantum theory of fields, there is an analogous pressure on every surface that can scatter light [1,2]. Even in the vacuum there are 'virtual photons' due to the quantum fluctuations of the electromagnetic field in its ground state. These virtual photons are scattered by any medium that can interact with light; this scattering exerts a force on the medium. Although quite small in magnitude, it has been measured experimentally [3].

In the original calculation of the Casimir force only simple shapes such as a sphere, or a rectangular slab were studied. Moreover, the medium was assumed to have ideal properties, such as perfect conductivity. Since that time, new methods have been developed, which allow the calculation of the Casimir energy for more general shapes. Of particular interest to us are the spectral methods [4–8]. (Methods that allow realistic computation of Casimir energy of micro-electromechanical or

E-mail address: rajeev@pas.rochester.edu

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MEM devices have been developed recently [9]; but we do not use them in this paper.) The basic idea behind these methods is a relationship (often called Levinson's theorem or Krein's trace formula) between the optical scattering matrix S and the density of states:

$$\rho(k) = \frac{1}{2\pi i}\frac{d}{dk}\,\mathrm{tr}\log S. \tag{1}$$

The Casimir energy is the sum over frequencies weighted by $\rho(k)$.

The physical argument that the Casimir force is due to the reflection of virtual photons suggests that it should be possible to express it entirely in terms of the probability of reflection. Unless the momentum of the photon changes during scattering, it should not contribute to the Casimir force.

In this paper we will show that the dispersion relations of scattering theory allow an answer for the density of states in terms of the reflection probability alone, in the one dimensional case. In the three dimensional case, there is a potential logarithmic divergence in $\mathrm{tr}\log S$. We show how to remove this by using a dispersion relation, without assuming rotational symmetry or using perturbation theory as in previous treatments [10]. An outcome is an 'anomaly' term in the density of states proportional to the integral of the potential. Contrary to the physical picture in terms of scattering of virtual photons, there is this one contribution to the Casimir effect that cannot be expressed in terms of the scattering matrix alone.

Much of our detailed analysis will be only carried out for the technically simpler case of a scalar field. As in traditional optical scattering theory, this scalar model gives a reasonable picture of the essential phenomenon. It should be straightforward although technically involved to extend our analysis to include polarization effects; i.e., scattering of vectorial fields. Moreover, we will ignore the effects of absorption of light: the effect of absorption on virtual photons needs a deeper analysis than we can provide at the moment. Also, we will consider only optical media that are time independent. For a technical reason, we will further assume that the scatterer is parity invariant; although it does not have to be symmetric otherwise.

We aim to give a more or less self-contained description, taking off from the discussion of scattering in standard textbooks. The erudite reader could skip ahead to Sections 3.1,3.2 and 5. No pretense at mathematical rigor is made. However, there will be an occasion for us to be careful about the definition of an infinite determinant (of the scattering operator) to avoid a logarithmic divergence.

2. Scattering of light

The propagation of light of (Ref. [11]) wave-number k in a dielectric medium is described by the equations

$$\nabla \times \mathbf{E} = ik\mathbf{H}, \quad \nabla \times \mathbf{H} = -ik\epsilon(x,k)\mathbf{E}. \tag{2}$$

Here $\epsilon(x,k)$ is the dielectric 'constant' which may in fact depend on position and on the wave-number k. (In other words, $\epsilon(x,k)$ is the square of the refractive index.) $\mathbf{E}$ and $\mathbf{H}$ are complex vector valued functions describing the amplitude of the electromagnetic wave.

Eliminating $\mathbf{H}$, we get

$$\nabla \times \nabla \times \mathbf{E}(x,k) = k^2\epsilon(x,k)\mathbf{E}. \tag{3}$$

In other words,

$$\nabla \times \nabla \times \mathbf{E}(x,k) + V(x)\mathbf{E} = k^2\mathbf{E}, \tag{4}$$

where the 'effective potential' [12] is

$$V(x) = k^2[1-\epsilon(x,k)]. \tag{5}$$