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PEPS as ground states: Degeneracy and topology

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ABSTRACT

We introduce a framework for characterizing Matrix Product States (MPS) and Projected Entangled Pair States (PEPS) in terms of symmetries. This allows us to understand how PEPS appear as ground states of local Hamiltonians with finitely degenerate ground states and to characterize the ground state subspace. Subsequently, we apply our framework to show how the topological properties of these ground states can be explained solely from the symmetry: We prove that ground states are locally indistinguishable and can be transformed into each other by acting on a restricted region, we explain the origin of the topological entropy, and we discuss how to renormalize these states based on their symmetries. Finally, we show how the anyonic character of excitations can be understood as a consequence of the underlying symmetries.

1. Introduction

What are the entanglement properties of quantum many-body states which characterize ground states of Hamiltonians with local interactions? The answer seems to be "an area law": the bipartite entanglement between any region and its complement grows as the area separating them – and not as their volume, as is the case for a random state (see [1] for a recent review). Moreover, particular corrections to this scaling law are linked with critical points (logarithmic corrections) or topological order (additive corrections). A rigorous general proof of the area law, however, could up to now only be given for the case of one-dimensional systems [2], where an area law has been proven for all systems with an energy gap above the ground state, whereas the currently strongest result for two

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dimensions [3,4] requires a hypothesis on the eigenvalue distribution of the Hamiltonian. Surprisingly, there is a completely general proof in arbitrary dimensions if instead, we consider the corresponding quantity for thermal states [5], and similar links to topological order persist [6].

The area law can be taken as a guideline for designing classes of quantum states which allow to faithfully approximate ground states of local Hamiltonians. There are several of these classes in the literature: Matrix Product States (MPS) [7] and Projected Entangled Pair States (PEPS) [8] are most directly motivated by the area law, but there are other approaches such as MERA (the Multi-Scale Entanglement Renormalization Ansatz) [9] which e.g. is based on the scale invariance of critical systems; all these classes are summarized under the name of Tensor Network or Tensor Product States. Though the main motivation to introduce them was numerical – they constitute variational ansatzes over which one minimizes the energy of a target Hamiltonians and thus obtains an approximate description of the ground state – they have turned out to be powerful tools for characterizing the role of entanglement in quantum many-body systems, and thus helped to improve our understanding of their physics.

In this paper, we are going to present a theoretical framework which allows us to understand how MPS and PEPS appear as ground states of local Hamiltonians, and to characterize the properties of their ground state subspace. This encompasses previously known results for MPS and particular instances of PEPS, while simultaneously giving rise to a range of new phenomena, in particular topological effects. Our work is motivated by the contrast between the rather complete understanding in one and the rather sparse picture in two dimensions, and we will review what is known in the following. We will thereby focus on analytical results, and refer the reader interested in numerical aspects to [10].

1.1. Matrix Product States

Matrix Product States (MPS) [7] form a family of one-dimensional quantum states whose description is inherently local, in the sense that the degree to which two spins can be correlated is related to their distance. The total amount of correlations across any cut is controlled by a parameter called the *bond dimension*, such that increasing the bond dimension allows to grow the set of states described. MPS have a long history, which was renewed in 1992 when two apparently independent papers appeared: In [11], Fannes et al. generalized the AKLT construction of [12] by introducing the so-called Finitely Correlated States, which in retrospect can be interpreted as MPS defined on an infinite chain; in fact, this work layed the basis for our understanding of MPS and introduced many techniques which later proved useful in characterizing MPS [13]. The other was [14], where White introduced the Density Matrix Renormalization Group (DMRG) algorithm, which can now be understood as a variational algorithm over the set of MPS. In [15], MPS were explained from a quantum information point of view by distributing "virtual" maximally entangled pairs between adjacent sites which can only be partially accessed by acting on the physical system. This entanglement-based perspective has since then fostered a wide variety of results.

1.1.1. The complexity of simulating one-dimensional systems

Motivated by the extreme success of DMRG, people investigated how hard or easy the problem of approximating the ground state of a 1D local Hamiltonian (or simply its energy) was. The history of this problem is full of interesting positive and negative results. A number of them was devoted to prove that every ground state of a gapped 1D local Hamiltonian can be approximated by an MPS [16,17]; this was finally proven by Hastings [2], justifying the use of MPS as the appropriate representation of the state of one-dimensional spin systems. Very recently, also in the positive, it was shown that dynamical programming could be used to find the best approximation to the ground state of a one-dimensional system within the set of MPS with fixed bond dimension in a provably efficient way [18,19]. On the other hand, in the negative it could be shown that finding the ground state energy of Hamiltonians whose ground states are MPS with a bond dimension polynomial in the system size is NP-hard [20]; this is based on a previous result of Aharonov et al. [21] proving that finding the ground state energy of 1D Hamiltonians is QMA-complete (the quantum version of NP-complete).