Annals of Physics 325 (2010) 2251–2306

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

Correlations of RMT characteristic polynomials and integrability: Hermitean matrices

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ARTICLE INFO

Article history:
Received 3 March 2010
Accepted 15 April 2010
Available online 19 August 2010

Keywords:
Random matrix theory
Characteristic polynomials
Spectral correlations
Integrable hierarchies
Bilinear identity
Virasoro constraints
Painlevé equations
Matrix integrals
Replica field theories
Replica limit

ABSTRACT

Integrable theory is formulated for correlation functions of characteristic polynomials associated with invariant non-Gaussian ensembles of Hermitean random matrices. By embedding the correlation functions of interest into a more general theory of τ functions, we (i) identify a zoo of hierarchical relations satisfied by τ functions in an abstract infinite-dimensional space and (ii) present a technology to translate these relations into hierarchically structured nonlinear differential equations describing the correlation functions of characteristic polynomials in the physical, spectral space. Implications of this formalism for fermionic, bosonic, and supersymmetric variations of zero-dimensional replica field theories are discussed at length. A particular emphasis is placed on the phenomenon of fermionic–bosonic factorisation of random-matrix-theory correlation functions.

1. Introduction

1.1. Motivation and definitions

Correlation functions of characteristic polynomials (CFCP) appear in various fields of mathematical and theoretical physics. (i) In quantum chaology, CFCP (i.a) provide a convenient way to describe the universal features of spectral statistics of a particle confined in a finite system exhibiting chaotic classical dynamics [9,4,49] and (i.b) facilitate calculations of a variety of

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0003-4916/$ - see front matter
doi:10.1016/j.aop.2010.04.005

important distribution functions whose generating functions may often be expressed in terms of CFCP (see, e.g. [3]). (ii) In the random matrix theory approach to quantum chromodynamics, CFCP allow to probe various QCD partition functions (see, e.g. [69]). (iii) In the number theory, CFCP have been successfully used to model behaviour of the Riemann zeta function along the critical line [41,42,36]. (iv) Recently, CFCP surfaced in the studies of random energy landscapes [27]. (v) For the rôle played by CFCP in the algebraic geometry, the reader is referred to the paper by Brézin and Hikami [13] and references therein.

In what follows, we adopt a formal setup which turns an $n \times n$ Hermitian matrix $\mathcal{H} = \mathcal{H}^\dagger$ into a central object of our study. For a *fixed* matrix $\mathcal{H}$, the characteristic polynomial $\det_n(\varsigma - \mathcal{H})$ contains complete information about the matrix spectrum. To study the *statistics* of spectral fluctuations in an *ensemble* of random matrices, it is convenient to introduce the correlation function $\Pi_{n|p}(\boldsymbol{\varsigma};\boldsymbol{\kappa})$ of characteristic polynomials

$$\Pi_{n|p}(\boldsymbol{\varsigma};\boldsymbol{\kappa}) = \left\langle \prod_{\alpha=1}^{p} \det{}_n^{\kappa_\alpha}(\varsigma_\alpha - \mathcal{H}) \right\rangle_{\mathcal{H}}. \tag{1.1}$$

Here, the vectors $\boldsymbol{\varsigma} = (\varsigma_1, \dots, \varsigma_p)$ and $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_p)$ accommodate the energy and the "replica" parameters, respectively. The angular brackets $\langle f(\mathcal{H}) \rangle_{\mathcal{H}}$ stand for the ensemble average

$$\langle f(\mathcal{H}) \rangle_{\mathcal{H}} = \int d\mu_n(\mathcal{H}) f(\mathcal{H}) \tag{1.2}$$

with respect to a proper probability measure

$$d\mu_n(\mathcal{H}) = P_n(\mathcal{H})(\mathcal{D}_n\mathcal{H}), \tag{1.3}$$

$$(\mathcal{D}_n\mathcal{H}) = \prod_{j=1}^{n} d\mathcal{H}_{jj} \prod_{j<k}^{n} d\Re\mathfrak{e}\mathcal{H}_{jk} d\Im\mathfrak{m}\mathcal{H}_{jk} \tag{1.4}$$

normalised to unity. Throughout the paper, the probability density function $P_n(\mathcal{H})$ is assumed to follow the trace-like law

$$P_n(\mathcal{H}) = \mathcal{C}_n^{-1} \exp[-\mathrm{tr}_n V(\mathcal{H})] \tag{1.5}$$

with $V(\mathcal{H})$ to be referred to as the confinement potential.

There exist two canonical ways to relate the spectral statistics of $\mathcal{H}$ encoded into the average p-point Green function

$$G_{n|p}(\boldsymbol{\varsigma}) = \left\langle \prod_{\alpha=1}^{p} \mathrm{tr}_n(\varsigma_\alpha - \mathcal{H})^{-1} \right\rangle_{\mathcal{H}} \tag{1.6}$$

to the correlation function $\Pi_{n|p}(\boldsymbol{\varsigma};\boldsymbol{\kappa})$ of characteristic polynomials.

- The supersymmetry-like prescription [21,67,31],

$$G_{n|p}(\boldsymbol{\varsigma}) = \left(\prod_{\alpha=1}^{p} \lim_{\varsigma'_\alpha \to \varsigma_\alpha} \frac{\partial}{\partial \varsigma_\alpha} \right) \Pi^{(\mathrm{susy})}_{n|p+p}(\boldsymbol{\varsigma}, \boldsymbol{\varsigma}'), \tag{1.7}$$

  makes use of the correlation function

$$\Pi^{(\mathrm{susy})}_{n|q+q'}(\boldsymbol{\varsigma}, \boldsymbol{\varsigma}') = \left\langle \prod_{\alpha=1}^{q} \det{}_n(\varsigma_\alpha - \mathcal{H}) \prod_{\beta=1}^{q'} \det{}_n^{-1}\left(\varsigma'_\beta - \mathcal{H}\right) \right\rangle_{\mathcal{H}} \tag{1.8}$$

  obtainable from $\Pi_{n|q+q'}(\boldsymbol{\varsigma}, \boldsymbol{\varsigma}'; \boldsymbol{\kappa}, \boldsymbol{\kappa}')$ by setting the replica parameters $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}'$ to the *integers* ± 1.
- On the contrary, the replica-like prescription [33,20],

$$G_{n|p}(\boldsymbol{\varsigma}) = \left(\prod_{\alpha=1}^{p} \lim_{\kappa_\alpha \to 0} \kappa_\alpha^{-1} \frac{\partial}{\partial \varsigma_\alpha} \right) \Pi_{n|p}(\boldsymbol{\varsigma}; \boldsymbol{\kappa}), \tag{1.9}$$