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Quantum mechanics in a two-dimensional spacetime: What is a wavefunction?

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A R T I C L E   I N F O

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A B S T R A C T

Conventional quantum mechanics specifies the mathematical properties of wavefunctions and relates them to physical experiments by invoking the Born postulate. There is no known direct connection between wavefunctions and any external physical object. However, in the case of a two-dimensional spacetime there is a completely classical context for wavefunctions where the connection with an external reality is transparent and unambiguous. By examining this case, we show how a classical stochastic process assembles a Dirac wavefunction based solely on the detailed counting of reversible paths. A direct comparison of how a related process assembles a Probability Density Function reveals both how and why PDFs and wavefunctions differ, including the ubiquitous implication of complex numbers for the latter. The appearance of wavefunctions in a context that is free of the complexities of quantum mechanics suggests the study of such models may shed some light on interpretive issues.

1. Introduction

The empirical accuracy of quantum mechanics makes the theory unsurpassed in the history of science. Despite this, there continue to be aspects of the theory that many scientists find controversial [1,2]. The practically universal agreement that *quantum mechanics provides a superb probability calculus* does not extend to questions involving the theory's interpretation. Opinions on interpretive issues cover a large spectrum.

Compare this situation with that of classical statistical mechanics and diffusion. In Table 1, six partial differential equations are listed. In the center column are three classical PDEs that describe

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Table 1
Three sets of partial differential equations are compared. The center column contains phenomenological equations that have a basis in kinetic theory. The PDF solutions are expected values of sums of the Bernoulli random variable. The right column contains 'quantum' equations obtained from the classical equations through a formal analytic continuation. We show that these equations also have a kinetic theory basis in which the solutions are expected values of sums of the Anti-Bernoulli random variable.

Kinetic 'picture'	Kac (Poisson)	Feynman chessboard
Telegraph/Dirac	$\frac{\partial \mathbf{U}}{\partial t} = -\boldsymbol{\sigma}_{\mathbf{z}} \frac{\partial U}{\partial z} + a\,\boldsymbol{\sigma}_{\mathbf{x}} U$	$\frac{\partial \Psi}{\partial t} = -\boldsymbol{\sigma}_{\mathbf{z}} \frac{\partial \Psi}{\partial z} + im\,\boldsymbol{\sigma}_{\mathbf{x}} \Psi$
Telegraph/KG	$\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial z^2} + a^2 U$	$\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial z^2} + (im)^2 \psi$
Heat/Schrödinger	$\frac{\partial U}{\partial t} = D \frac{\partial^2 U}{\partial x^2}$	$\frac{\partial \psi}{\partial t} = iD \frac{\partial^2 \psi}{\partial x^2}$
Characteristic random variable	Bernoulli $X \in \{1, 0\}$	Anti-Bernoulli $Y \in \{1, 0, -1\}$

diffusive processes. Their solutions are typically probability density functions that are obtained by counting paths with a random variable that detects the presence or absence of a path. The random variable X in this case is Bernoulli:

$$X = \begin{cases} 1 & \text{path link present} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

the stochastic version of an indicator function. Ultimately, the PDF solutions are continuum limits of the expected values of sums of the Bernoulli random variable. That is:

$$U(x,t) = E\left[\sum_{\text{Path Ensemble}} X\right]. \tag{2}$$

The solutions can be treated as probability densities since the sums of the Bernoulli random variable are non-negative and continuity of the paths allow U to be normalizable as a PDF in the continuum limit. Indeed, (2) is an expression of the frequency-based picture of probability.

On the right of the table we see, respectively, the Dirac, Klein–Gordon and Schrödinger equations. Each of these may be obtained from the classical equation in the same row by the conversion of a single real constant to an imaginary constant. However, the solutions of these equations are wavefunctions, not PDFs. The presence of the imaginary constant removes the solutions from the domain of functions that would satisfy the properties of a PDF. Although the classical equations on the left are ultimately phenomenological with a basis in kinetic theory, the equations on the right are regarded as fundamental with no prior basis in an underlying microscopic model.

Comparing the two sets of equations, we understand the classical equations well enough to see how they arise from elementary properties of small classical particles in random motion. If we ask the question "What is a probability density function?" in the context of the solutions of these equations, we get a precise answer that is transparent with little need for 'interpretation'.

The purpose of this article is to show that we can do the same for the quantum equations and associated wavefunctions in a two-dimensional spacetime provided particle paths treat both dimensions as spacelike in a manner that we shall describe shortly. The result is interesting in a number of ways. The model we discuss provides a simple classical model that can, in principle, be used to quantitatively simulate single-particle quantum mechanics in one dimension. Hints that this may be extended to three dimensions exist [3,4] and will be confirmed in a future work. In addition to this, wavefunctions appear here as natural generalizations of PDFs to include counting processes for reversing paths. As such they may be studied as stochastic processes independently of their context in quantum mechanics. Finally, by comparing the classical and quantum contexts we anticipate that interpretative issues about quantum mechanics [1] may be brought into sharp focus.

2. Kac's model and PDFs

We begin by reviewing a version of Kac's model for the Telegraph equations [5]. Consider walks taking place on a lattice in the x–y plane. Particles traverse diagonal links on the lattice moving in