Annals of Physics 324 (2009) 1340–1359

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

A unified theory of quantum holonomies

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ARTICLE INFO

Article history:
Received 23 February 2009
Accepted 14 March 2009
Available online 24 March 2009

PACS:
03.65.Vf
03.65.Ca
42.50.Dv

Keywords:
Geometric phase
Exotic hononomies
Gauge theory

ABSTRACT

A periodic change of slow environmental parameters of a quantum system induces quantum holonomy. The phase holonomy is a well-known example. Another is a more exotic kind that exhibits eigenvalue and eigenspace holonomies. We introduce a theoretical formulation that describes the phase and eigenspace holonomies on an equal footing. The key concept of the theory is a gauge connection for an ordered basis, which is conceptually distinct from Mead–Truhlar–Berry's connection and its Wilczek–Zee extension. A gauge invariant treatment of eigenspace holonomy based on Fujikawa's formalism is developed. Example of adiabatic quantum holonomy, including the exotic kind with spectral degeneracy, are shown.

1. Introduction

Consider a quantum system in a stationary state. Let us adiabatically change a parameter of the system along a closed path where the spectral degeneracy is assumed to be absent. We ask the destination of the state after a change of the parameter along the path. This question is frequently raised in discussions of the Berry phase [1]. An answer, which is widely shared since Berry's work, is that a discrepancy remains in the phase of the state vector, even after the dynamical phase is excluded. Indeed this is correct in a huge amount of examples [2,3]. However, it is shown that this answer is not universal in a recent report of exotic anholonomies [4] in which the initial and the final states are orthogonal in spite of the absence of the spectral degeneracy. In other words, the eigenspace

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0003-4916/$ - see front matter
doi:10.1016/j.aop.2009.03.006

associated with the adiabatic cyclic evolution exhibits discrepancy, or anholonomy. Furthermore, the eigenspace discrepancy induces another discrepancy in the corresponding eigenenergy.

For the phase discrepancy, an established interpretation in terms of differential geometry allows us to call it the phase *holonomy* [5]. This interpretation naturally invites its non-Abelian extension, which has been subsequently discovered by Wilczek and Zee in systems with spectral degeneracies [6]. Contrary to this, any successful association of the eigenspace discrepancy with the concept of holonomy has not been known.

The aim of this paper is to demonstrate that an interpretation of the eigenspace discrepancy in terms of holonomy is indeed possible. To achieve this, we introduce a framework that treats the phase and the eigenspace holonomies in a unified manner in Section 2. The key concept is a non-Abelian gauge connection that is associated with a parameterized basis [7], and the identification of the place where the gauge connection resides in the time evolution. This is achieved through a fully gauge invariant extension of Fujikawa's formulation that has been introduced for the phase holonomy [8,9]. Our approach is illustrated by the analysis of adiabatic quantum holonomies of three examples. First, Berry's Hamiltonian with spin-$\frac{1}{2}$ is revisited in Section 3. The role of parallel transport [10,5], which accompanies the multiple-valuedness of a parameterized basis, in our formulation will be emphasized. The second example, shown in Section 4, exhibits exotic holonomies without spectral degeneracy. The last example, shown in Section 5, is the simplest examples of the exotic holonomies in the presence of degeneracy, i.e., the eigenspace holonomy á la Wilczek and Zee. Section 6 provides a summary and an outlook. A brief, partial report of the present result can be found in Ref. [11].

2. A gauge theory for a parameterized basis

Two building blocks of our theory, a gauge connection that is associated with a parameterized basis [7], and Fujikawa formalism, originally conceived for the phase holonomy, are presented in order to introduce our approach to quantum holonomies.

2.1. A gauge connection

In the presence of the quantum holonomy, basis vectors are, in general, multiple-valued as functions of a parameter. In order to cope with such multiple-valuedness, we introduce a gauge connection for a parameterized basis. This has been introduced by Filipp and Sjöqvist [7] to examine Manini–Pistolesi off-diagonal geometric phase [12]. As is explained below, this gauge connection is different from Mead–Truhlar–Berry's [13,1] and Wilczek–Zee's gauge connections [6], which describe solely the phase holonomy.

For N-dimensional Hilbert space $\mathscr{H}$, let $\{|\xi_n(s)\rangle\}_{n=0}^{N-1}$ be a complete orthogonal normalized system that is smoothly depends on a parameter s. The parametric dependence induces a gauge connection $A(s)$, which is a $N \times N$ Hermite matrix and whose (n,m)-th element is

$$A_{nm}(s) \equiv i\langle\xi_n(s)|\frac{\partial}{\partial s}|\xi_m(s)\rangle. \tag{1}$$

By definition, $A(s)$ is non-Abelian. For given $A(s)$, the basis vector $|\xi_n(s)\rangle$ obeys the following differential equation

$$i\frac{\partial}{\partial s}|\xi_m(s)\rangle = \sum_n A_{nm}(s)|\xi_n(s)\rangle \tag{2}$$

and we may solve this equation with an "initial condition" at $s = s'$.

The dynamical variable of the equation of motion (2) is an ordered sequence of basis vectors, also called a frame,

$$f(s) \equiv [\,|\xi_0(s)\rangle, \quad |\xi_1(s)\rangle, \quad \ldots, \quad |\xi_{N-1}(s)\rangle\,]. \tag{3}$$