C. R. Physique 16 (2015) 723–728

Contents lists available at ScienceDirect

Comptes Rendus Physique

www.sciencedirect.com

Prix Langevin de l'Académie des sciences

# Landau damping and inhomogeneous reference states

## *Amortissement Landau et états de référence non homogènes*

Julien Barré [a,*], Alain Olivetti [a], Yoshiyuki Y. Yamaguchi [b]

[a] *Laboratoire Jean-Alexandre-Dieudonné, Université de Nice–Sophia Antipolis, UMR CNRS 7351, parc Valrose, 06108 Nice cedex 02, France*
[b] *Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan*

A R T I C L E   I N F O

*Article history:*
Available online 18 April 2015

*Keywords:*
Landau damping
Plasma physics
Fluid dynamics
Astrophysics
Vlasov equation
Inhomogeneous stationary state

*Mots-clés :*
Amortissement de Landau
Physique des plasmas
Dynamique des fluides
Astrophysique
Équation de Vlasov
État stationnaire non homogène

A B S T R A C T

Landau damping is a fundamental phenomenon in plasma physics, which also plays an important role in astrophysics, and sometimes under different names, in fluid dynamics, and other fields. Its theoretical discussion in the framework of the Vlasov equation often assumes that the reference stationary state is homogeneous in space. However, Landau damping around an inhomogeneous reference stationary state, a natural setting in astrophysics for instance, induces new mathematical difficulties and physical phenomena. The goal of this article is to provide an introduction to these problems and the questions they raise.


R É S U M É

L'amortissement Landau est fondamental en physique des plasmas, et joue, parfois sous un nom différent, un rôle important en astrophysique, en dynamique des fluides et dans d'autres domaines. Son traitement théorique à partir de l'équation de Vlasov suppose souvent que l'état stationnaire de référence est homogène en espace. Néanmoins, un état stationnaire non homogène en espace, un cadre naturel en astrophysique par exemple, induit des difficultés mathématiques et des phénomènes physiques nouveaux. Le but de cet article est de fournir une introduction à ces problèmes et aux questions qu'ils soulèvent.


## 1. Introduction

Landau damping was first introduced almost 70 years ago [1], and it has since then been an active subject of research, whose importance goes far beyond its original realm of the Vlasov equation in plasma physics. The purpose of this article is to provide an introduction to the specificities of Landau damping close to a stationary state of the Vlasov equation which is not homogeneous in space: we describe in particular the various singularities governing the behavior of a perturbation (Table 1), and how they yield a two-step relaxation, see Fig. 3. This situation is in particular relevant for astrophysical applications of the Vlasov equation, but it is striking that a phenomenology similar to the one we will describe here is

* Corresponding author.
*E-mail address:* Julien.BARRE@unice.fr (J. Barré).

http://dx.doi.org/10.1016/j.crhy.2015.03.013

encountered in very different physical systems: fluids described by Euler equation [2], Kuramoto model for oscillators synchronization [3], sound propagation in bubbly fluids [4]... For self-consistency, we will first recall some basic facts about the Vlasov equation and standard Landau damping as it was first introduced in plasma physics, close to a homogeneous-in-space stationary state.

## 2. The Vlasov equation

When one tries to describe the positions and velocities of a large number of interacting particles by a phase space density, different situations may occur: if the interactions between particles are strong and rare, so that they involve only two particles each time, one expects a Boltzmann-like equation; if on the contrary one particle feels the effect of many others, the individual effect of each one being weak, one enters the realm of the Vlasov equation. This scaling limit, sometimes called mean-field limit, appears in many areas of physics, the two main examples being the particles in Coulombian interaction, and the particles in Newtonian interaction. In the plasma physics context, it was indeed introduced by Vlasov [5]; in the astrophysical context, the Vlasov equation is usually called "collisionless Boltzmann equation". We write down the equation in its Hamiltonian form:

$$\partial_t f + \{f, h\} = 0 \tag{1}$$

where $f$ is the phase space density, normalized to 1, $\{f, g\}$ is the Poisson bracket

$$\{f, g\} = \frac{1}{M}\left(\nabla_{\mathbf{x}} f \cdot \nabla_{\mathbf{v}} g - \nabla_{\mathbf{v}} f \cdot \nabla_{\mathbf{x}} g\right) \tag{2}$$

$M$ is the particles' mass and $h(\mathbf{x}, \mathbf{v}, t)$ the Hamiltonian

$$h(\mathbf{x}, \mathbf{v}, t) = \frac{M\mathbf{v}^2}{2} + \phi(\mathbf{x}, t) \tag{3}$$

The potential $\phi(\mathbf{x}, t)$ is created by $f$. For Coulomb or Newton interaction, $\phi$ is related to the spatial density $\rho = \int f \mathrm{d}\mathbf{v}$ through the Poisson equation:

$$\Delta\phi = C\rho \tag{4}$$

where $C = 4\pi G$ for the Newton interaction, with $G$ the gravitational constant, and $-4\pi e^2/M$ for the Coulomb one. More generally, $\phi$ may be given by the convolution of the density $\rho$ and a kernel $K$:

$$\phi(\mathbf{x}) = \int K(\mathbf{x} - \mathbf{y})\rho(\mathbf{y})\,\mathrm{d}\mathbf{y} \tag{5}$$

For instance, $K$ is proportional to $1/|\mathbf{x}|$ for Coulomb (resp. $-1/|\mathbf{x}|$ for Newton) interaction in 3D. Physically, the Vlasov equation is simply a transport equation, meaning that each particle moves in the field that all particles create. Hereafter, we set $M = 1$ for simplicity.

The convergence of the discrete dynamics of $N$ particles towards the continuous Vlasov equation has been rigorously proved [6,7], but under the hypothesis that the kernel $K$ is regular enough. In particular, this excludes the Coulomb and Newton cases!

## 3. Stationary states and their stability

The Vlasov equation (1) admits an infinite number of stationary states. Indeed, any function $f = \varphi(h(\mathbf{x}, \mathbf{v}))$ that depends on the phase space coordinates only through the Hamiltonian $h$ is a stationary solution. An easy special case is provided by a function $f$ that depends only on $\mathbf{v}^2$ (this implies that $f$ is homogeneous in space), with a vanishing self-consistent potential. Space inhomogeneous stationary solutions, needed for instance to describe self-gravitating systems, are more difficult to construct, since one needs to solve the self-consistent equation (4) for the potential.

Studying the Vlasov dynamics close to a stationary solution proved a difficult endeavor, with surprises even at the linear level. When the linearized Vlasov operator has an eigenvalue with positive real part, the corresponding stationary state is clearly unstable. Furthermore, since the Vlasov equation has a Hamiltonian structure, it is easy to realize that the spectrum of the linearized operator is symmetric with respect to the imaginary axis; hence, if there is an eigenvalue with a negative real part, there is also one with a positive real part, and the corresponding stationary state is also unstable. Thus, the stable stationary states are only marginally stable, the spectrum being included in the imaginary axis. Yet, a perturbation around a stable stationary state may decay in some sense exponentially, with a well-defined rate: this surprising exponential decay for a time-reversible equation is the well-known Landau damping phenomenon, first described in 1946 [1]. In the following, we consider such damping of a perturbation $\delta f$ around a stable stationary state $f_0$.