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Inner structure of $Spin^c(4)$ gauge potential on 4-dimensional manifolds

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ABSTRACT

The decomposition of $Spin^c(4)$ gauge potential in terms of the Dirac 4-spinor is investigated, where an important characterizing equation $\Delta A_\mu = -\lambda A_\mu$ has been discovered. Here, λ is the vacuum expectation value of the spinor field, $\lambda = \|\Phi\|^2$, and A_μ the twisting $U(1)$ potential. It is found that when λ takes constant values, the characterizing equation becomes an eigenvalue problem of the Laplacian operator. It provides a revenue to determine the modulus of the spinor field by using the Laplacian spectral theory. The above study could be useful in determining the spinor field and twisting potential in the Seiberg–Witten equations. Moreover, topological characteristic numbers of instantons in the self-dual sub-space are also discussed.

1. Introduction

Great importance has been attached to four-dimensional manifolds in the research of topology [1]. In 1983, Donaldson used instantons to prove new theorems and developed topological invariants for 4-manifolds. In 1994, Witten proposed the Seiberg–Witten equations in the study of the $Spin^c(4)$ bundle and gave a new invariant for classifying 4-manifolds. In this paper, we will use a physical point of view—gauge potential decomposition—to study the gauge fields and topology on 4-manifolds.

In recent years, gauge potential decomposition has established itself a useful tool in physicists' mathematical arsenal [2–5]. Its main idea is to reparametrize gauge potentials, such that characteristic classes—which are originally conveyed by gauge field strengths—can be re-expressed in terms of basic

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fields on manifolds. These basic fields can be vector fields, spinor fields, etc. It is known that characteristic classes are topological invariants causing non-trivial observable topological effects in physical systems. Such a change of building blocks will lead to disclosure of hidden geometric aspects of physical problems, such as topological charges and locations of excitations, etc., which are difficult to achieve by other means. This is the significance of gauge potential decomposition, consistent with the essence of the Poincaré–Hopf theorem [6].

In this paper, our starting point is the decomposition of the $Spin^c(4)$ gauge potential. The decomposing basic field employed is the Dirac spinor field. The results could be useful for the discussion of the Seiberg–Witten (SW) equations. This paper is arranged as follows. In Section 2, the decompositions of the $Spin^c(4)$ potential is given. In Section 3, an important characterizing equation is obtained. It is found that this equation will become an eigenvalue problem of the Laplacian operator when the vacuum expectation value of the spinor field takes constant values. This study could be applied in determining the twisting potential and spinor field in the SW equations. In Section 4, we investigate the topological characteristic numbers of instantons in the self-dual $SU(2)_+$ sub-space. In Section 5, we conclude the paper by presenting a summary and discussion.

2. Decomposition of $Spin^c(4)$ Gauge potential

Let $\mathcal{M}$ be an oriented closed Riemannian 4-manifold possessing a $Spin(4)$-structure, $Spin(4) = SU(2)_+ \otimes SU(2)_-$ [1]. Let P be a principal $Spin^c(4)$-bundle on $\mathcal{M}$. $Spin^c(4)$ is obtained by $U(1)$-twisting $Spin(4), Spin^c(4) = Spin(4) \otimes L$, where L is the twisting line bundle. $Spin(4)$ is the double-cover of the group space of $SO(4)$. Their Lie algebras $\mathfrak{spin}(4)$ and $\mathfrak{so}(4)$ are isomorphic.

Consider a general massless Dirac equation with an electromagnetic field,

$$\gamma^\mu \partial_\mu \Psi - \gamma^\mu \omega_\mu \Psi - i\gamma^\mu A_\mu \Psi = 0. \tag{1}$$

Here, $\mu = 1, 2, 3, 4$ denotes the base manifold indices. $D_\mu \Psi$ is the covariant derivative for the Dirac spinor field $\Psi, D_\mu = \partial_\mu - (\omega_\mu + iA_\mu)$. $(\omega_\mu + iA_\mu)$ is the $Spin^c(4)$ gauge potential. A_μ is the $U(1)$ potential of $L, A_\mu \in \mathbb{R}$. ω_μ is the $SO(4)$ potential, $\omega_\mu = \frac{1}{2}\omega_{\mu ab} I_{ab}$, with $\omega_{\mu ab} = -\omega_{\mu ba}$. $I_{ab} (a \neq b)$ is the $SO(4)$ generator realized by the Clifford algebraic 2-vector, $I_{ab} = \frac{1}{4}[\gamma_a, \gamma_b] = \frac{1}{2}\gamma_a \gamma_b$, with $a, b = 1, 2, 3, 4$ denoting the Clifford algebraic indices. $\gamma^\mu = e^\mu_a \gamma_a$ is the general Gamma-matrices, raised by the vierbein e^μ_a from the Dirac matrices γ_a.

Our task is to decompose the $Spin^c(4)$ potential $\omega_\mu + iA_\mu$ in terms of the basic field Ψ. Defining $\omega_{abc} = e^\mu_a \omega_{\mu bc}$, one has

$$\gamma_a \partial_a \Psi - \frac{1}{4}\omega_{abc}\gamma_a \gamma_b \gamma_c \Psi - iA_a \gamma_a \Psi = 0, \tag{2}$$

where $\partial_a = e^\mu_a \partial_\mu$ and $A_a = e^\mu_a A_\mu$. We notice that ω_{abc} may be written in three parts:

$$\omega_{abc} = \omega^A_{abc} + \omega^{S_1}_{abc} + \omega^{S_2}_{abc}, \tag{3}$$

where ω^A_{abc} is fully anti-symmetric for $abc, \omega^{S_1}_{abc}$ symmetric for ab, and $\omega^{S_2}_{abc}$ symmetric for ac:

$$\omega^A_{abc} = \frac{1}{3}(\omega_{abc} + \omega_{bca} + \omega_{cab}), \quad \omega^{S_1}_{abc} = \frac{1}{3}(\omega_{abc} + \omega_{bac}), \quad \omega^{S_2}_{abc} = \frac{1}{3}(\omega_{abc} + \omega_{cba}). \tag{4}$$

Defining for convenience

$$\bar{\omega}_b = 2\omega_{aba}, \quad \tilde{\omega}_a = \epsilon_{abcd}\omega^A_{bcd}, \tag{5}$$

one obtains after simple Clifford algebra:

$$\omega^A_{abc}\gamma_a\gamma_b\gamma_c = -\gamma_d\gamma_5\tilde{\omega}_d, \quad \omega^{S_1}_{abc}\gamma_a\gamma_b\gamma_c = -\frac{1}{3}\bar{\omega}_c\gamma_c, \quad \omega^{S_2}_{abc}\gamma_a\gamma_b\gamma_c = -\frac{2}{3}\bar{\omega}_b\gamma_b, \tag{6}$$

where $\gamma_5 = -\gamma_1\gamma_2\gamma_3\gamma_4$. Hence (2)can be rewritten as

$$\gamma_a \partial_a \Psi + \frac{1}{4}\gamma_a(\bar{\omega}_a + \gamma_5\tilde{\omega}_a)\Psi - iA_a\gamma_a\Psi = 0. \tag{7}$$