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Annals of Physics 321 (2006) 318–330

ANNALS of PHYSICS

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Hamilton–Jacobi theory for Hamiltonian systems with non-canonical symplectic structures

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Received 13 April 2005; accepted 19 August 2005
Available online 28 September 2005

Abstract

A proposal for the Hamilton–Jacobi theory in the context of the covariant formulation of Hamiltonian systems is done. The current approach consists in applying Dirac's method to the corresponding action which implies the inclusion of second-class constraints in the formalism which are handled using the procedure of Rothe and Scholtz recently reported. The current method is applied to the non-relativistic two-dimensional isotropic harmonic oscillator employing the various symplectic structures for this dynamical system recently reported.

PACS: 45.20.Jj

Keywords: Hamilton–Jacobi theory; Hamiltonian systems; Constrained systems

1. Introduction

To set down the issue analyzed in this paper, we begin first with a brief discussion of the standard treatment of Hamiltonian systems and after that with a brief summary of what we call a genuine covariant description of Hamiltonian dynamics.

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doi:10.1016/j.aop.2005.08.008

1.1. Canonical formulation of Hamiltonian systems

In the standard treatment of Hamiltonian dynamics, the equations of motion are written in the form [1]

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad i = 1, 2, \ldots, n, \tag{1}$$

where H is the Hamiltonian of the system, the variables (q^i, p_i) are canonically conjugate to each other in the sense that

$$\{q^i, q^j\} = 0, \quad \{q^i, p_j\} = \delta^i_j, \quad \{p_i, p_j\} = 0, \tag{2}$$

where $\{,\}$ is the Poisson bracket defined by (summation convention is used)

$$\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}. \tag{3}$$

1.2. Covariant formulation of Hamiltonian systems

The symplectic geometry involved in the Hamiltonian description of mechanics can clearly be appreciated if Eq. (1) are written in the form

$$\dot{x}^\mu = \omega^{\mu\nu}\frac{\partial H}{\partial x^\nu}, \tag{4}$$

with $(x^\mu) = (q^1, \ldots, q^n; \quad p_1, \ldots, p_n)$ and

$$(\omega^{\mu\nu}) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{5}$$

where 0 is the zero $n \times n$ matrix and I is the unity $n \times n$ matrix. Moreover, Eq. (3) acquires the form

$$\{f, g\} = \frac{\partial f}{\partial x^\mu}\omega^{\mu\nu}\frac{\partial g}{\partial x^\nu}, \tag{6}$$

from which Eq. (2) can be rewritten as

$$\{x^\mu, x^\nu\} = \omega^{\mu\nu}. \tag{7}$$

From this viewpoint, the coordinates (x^μ) locally label the points x of the phase space Γ associated to the dynamical system on which the symplectic structure $\omega = \frac{1}{2}\omega_{\mu\nu}(x)\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ is defined. The two-form ω is closed, i.e., $\mathrm{d}\omega = 0$ which is equivalent to the fact that the Poisson bracket satisfies the Jacobi identity [1]. Also ω is non-degenerate, i.e., $\omega_{\mu\nu}v^\nu = 0$ implies $v^\mu = 0$ which means that there exists the inverse matrix $(\omega^{\mu\nu})$. The equations of motion of Eq. (4) are *covariant* in the sense that they maintain their form if the canonical coordinates are replaced by a completely arbitrary set of coordinates in terms of which $(\omega^{\mu\nu})$ need not be given by Eq. (5). Similarly, it is possible to retain the original coordinates (q^i, p_i) and still write the original equations of motion (1) in the Hamiltonian form (4), but now employing alternative symplectic structures $\omega^{\mu\nu}(x)$, distinct to that given in Eq. (5), and taking as Hamiltonian any real function on Γ which is a constant of motion for the system. This means that the writing of the equations of motion of a dynamical system in Hamiltonian form is *not* unique [2–7]. More precisely, from the covar-