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# Parametric resonance of intrinsic localized modes in coupled cantilever arrays

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ABSTRACT

In this study, the parametric resonances of pinned intrinsic localized modes (ILMs) were investigated by computing the unstable regions in parameter space consisting of parametric excitation amplitude and frequency. In the unstable regions, the pinned ILMs were observed to lose stability and begin to fluctuate. A nonlinear Klein–Gordon, Fermi–Pasta–Ulam-like, and mixed lattices were investigated. The pinned ILMs, particularly in the mixed lattice, were destabilized by parametric resonances, which were determined by comparing the shapes of the unstable regions with those in the Mathieu differential equation. In addition, traveling ILMs could be generated by parametric excitation.



## 1. Introduction

Intrinsic localized modes (ILMs), which are spatially localized vibrations in nonlinear lattices, have attracted much attention since they were first predicted in an anharmonic lattice by Sievers and Takeno [1]. At present, ILMs are employed in diverse physical and artificial systems [2]. Micro-cantilever arrays fabricated from microelectromechanical system (MEMS) technology are among the artificial nonlinear lattices designed to have ILMs [3,4]. The observation and manipulation of ILMs in micro-cantilever arrays triggered application-based research. The keys to realizing ILM applications are constructing, controlling, and deconstructing ILMs as desired. It has already been shown that pinned ILMs in micro-cantilever arrays can be manipulated in steps [4]. In addition, moving ILMs, which travel with almost constant speeds, have been generated from pinned ILMs by adjusting their driving frequencies [5]. Thus, ILMs can be controlled without spatial decay. In addition, moving ILMs are identified in NaCl crystals by Dmitriev et al. [6]. It strongly suggests that the ILM will play a crucial role in nanotechnology.

In our previous research, another method for manipulating ILMs in micro-cantilever arrays was proposed, called capture and release manipulation [7]. In this method, a nonlinear coupling coefficient is varied to manipulate the ILM stability. An initially stable ILM begins to wander the lattice as it becomes unstable. The wandering ILM can be pinned again if the adjusting parameter is returned to its initial value at the appropriate time. The stability change used for the manipulation is due to the saddle-node bifurcations of ILMs [8]. Another means of adjusting ILM stability is through parametric excitation. For discrete nonlinear Schrödinger equation, it has been shown that the stability of ILM, which is also called discrete soliton, is changed with respect to the strength of parametric excitation [9,10]. Kenig et al. have also shown that the creation, stability, and interaction of ILMs in a parametrically driven coupled MEMS resonators having nonlinear damping [11]. Cuevas et al. demonstrated that the stability of an ILM in a forced-damped array of coupled pendula changed when the parametric excitation amplitude was varied [12]. They also succeeded in creating a traveling ILM in the same manner. Therefore, it is important to understand how ILMs depend upon temporary varied parameters for realization of control of ILMs.

The objective of this study was to generate a traveling ILM using parametric excitation. Unstable regions in which the ILMs lost stability and began to fluctuate along the lattice were investigated with respect to parametric excitation frequency and amplitude for the three different types of micro-cantilever arrays. The unstable regions were compared with those in the Mathieu differential equation for confirming the fact that the destabilization is caused by the parametric resonance of ILM.

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## 2. Model

Intrinsic localized modes in a coupled cantilever array can be analyzed as those in a nonlinear coupled oscillator with a viscous damping and an external driver [3,4]. Although the effect of the damping and the driving force to the motion of cantilevers should be considered for more precise analysis [13,14], we focus on a simple coupled nonlinear oscillator array, in which there are nonlinearities in both the on-site and inter-site potentials, because the viscous damping will be sufficiently small as long as the coupled cantilever array is placed in high vacuum environment [3,4]. It can be expected that the behavior of ILM will be qualitatively similar to that in the real system [2,15]. The equation of motion without the damping and the driving force is given by

$$\begin{aligned}\ddot{u}_n &= -u_n - \alpha(u_n - u_{n+1}) - \alpha(u_n - u_{n-1}) \\ &\quad - \beta_1 u_n^3 - \beta_2(u_n - u_{n+1})^3 - \beta_2(u_n - u_{n-1})^3 \\ &\quad (n = 1,\ 2,\ \cdots,\ N),\end{aligned} \tag{1}$$

where $u_n$ denotes the displacement of the $n$th cantilever from the equilibrium position and all of the coefficients are assumed to be positive. In this study, the linear coupling coefficient $\alpha$ was fixed at 0.1 and the total number of cantilever is set at $N = 8$. The boundary condition is assumed to be the periodic boundary condition for eliminating the effect of breaking the translational symmetry [8]. The ratio of the nonlinear coupling coefficient $\beta_2$ to the on-site nonlinearity $\beta_1$ plays a crucial role in ILM stability [7,8]. If $\beta_r = \beta_2/\beta_1$ is chosen at zero, the equation of motion coincides with a nonlinear Klein–Gordon (NKG)-type lattice. Nonlinear oscillators which have cubic nonlinearity in the restoring force are linearly coupled. In this lattice, site-centered modes are stable, whereas bond-centered modes are unstable [8]. In contrast, Eq. (1) becomes a Fermi–Pasta–Ulam (FPU)-like lattice when the ratio increases toward infinity, namely, when $\beta_r \to \infty$. Linear oscillators are nonlinearly coupled. For the FPU-like lattice, the site-centered modes are unstable, whereas the bond-centered modes are stable [8]. Equation (1) switches these two types of lattices to around $\beta_r = 0.5453$ if the total energy and the number of oscillators are set at 2.5 and 8, respectively [8]. In this investigation, three lattice types were studied: a NKG lattice ($\beta_1 = 1,\ \beta_2 = 0$), FPU-like lattice ($\beta_1 = 0,\ \beta_2 = 1$), and mixed lattice ($\beta_1 = 2,\ \beta_2 = 1$).

## 3. Stability and Floquet multipliers

Two ILM types are shown in Fig. 1(a). The left side is referred to as the site-centered or Sievers–Takeno (ST) mode, whereas the other is called bond-centered or Page (P) mode [16]. Since ILMs are time-periodic solutions, they can be treated as a fixed point of a Poincaré map *i.e.* $x^* = P(x^*)$. A small perturbation to the fixed point gives a linearized equation $y_{k+1} = DP(x^*)y_k$, where $k$ is positive integer and $DP(x^*)$ is Jacobian matrix at $x^*$. Therefore the stability of the fixed point, namely, the ILM can be determined by eigenvalues of the Jacobian matrix, which are called characteristic or Floquet multipliers [2]. In the left side of Fig. 1(b), the characteristic multipliers of a stable ST mode in a mixed lattice are shown. All of the multipliers are on the unit circle. In this case, the P mode is unstable (see the right side of Fig. 1(b)). It is known that their stability are flipped when $\beta_r \simeq 0.545358$ for $H = 2.5$ [8], as well as in the nonlinear Schrödinger equation [17].

The characteristic multipliers can be classified based on the spatial distributions of their corresponding eigenvectors. For the case shown in Fig. 1(b), $\rho_1$, $\rho_2$, $\rho_3$, and $\rho_4$ have spatially localized eigenvectors, whereas the other multipliers have spatially extended eigenvectors. Since Eq. (1) is a Hamiltonian system, two characteristic multipliers are located at $+1$ in the complex plane. These multipliers are considered the phase and growth modes [2,18]. Perturbing the phase mode changes the overall ILM phase, and the growth mode changes the total energy of the ILM. Another eigenvalue having the spatially localized eigenvector $\rho_3$ is called the pinning or translational mode [2,18]. The spatial symmetry of any particular eigenvector is opposite that of the ILM. Thus, a perturbation along the translational mode changes the ILM position along the lattice.

If a fixed point of a Poincaré map corresponding an ILM in the original equation is perturbed along an eigenvector $p_{\rho_i}$, the perturbed solution $y_k = x^* + \epsilon p_{\rho_i}$ will oscillate with frequency $f_i = \arg \rho_i / 2\pi$. If $f_i$ is a rational number $m/n$, the perturbed solution returns to the original position by mapping $n$ times, namely, $y_k = DP(x^*)^n y_k$. In original equation, the perturbation will cause a small fluctuation around the trajectory of ILM. Since the frequency of the ILM is $\omega_{\mathrm{ILM}}$, the fluctuation will have the frequency of $n\omega_{\mathrm{ILM}}$. Therefore, frequency of an oscillation caused by a small perturbation along the eigenspace spanned by $\rho_i$ and $\bar{\rho}_i$ is given by

$$\Omega_i = \frac{\arg \rho_i}{2\pi}\omega_{\mathrm{ILM}}, \tag{2}$$

where $\omega_{\mathrm{ILM}}$ is the angular frequency of the ILM. If the fluctuation is along the translational mode, the perturbed ILM will oscillate with an angular frequency $\Omega_3$. Therefore, the motion of the ILM fluctuation can be written as approximately

$$\ddot{\xi} = -\Omega_3^2 \xi, \tag{3}$$

where $\xi$ represents the displacement from the original position of the ILM, $X_{\mathrm{ILM}}$. Therefore, the absolute position of the fluctuating ILM is given by $X = \xi + X_{\mathrm{ILM}}$, where $X_{\mathrm{ILM}} \in [1/2, N + 1/2)$ is an integer for ST mode or a half-integer for P mode.

## 4. Parametric resonance

The frequencies of the fluctuations depend on the coefficient $\beta_2$ as shown in Fig. 2, that is $\Omega_i = \Omega_i(\beta_2)$. Let $\beta_2$ be a time-periodic function $\beta_2(t) = \beta_2 + \epsilon \sin \nu t$, where $\epsilon$ is positive and sufficiently small. By transforming the time variable $t \to t/\nu$ and considering the first order of $\epsilon$ of a Taylor expansion of Eq. (3), Eq. (3) becomes the Mathieu equation:

$$\begin{aligned}\ddot{\xi} &= -\frac{1}{\nu^2}\Omega_3^2(\beta_2 + \epsilon \sin t)\xi \\ &\sim -\frac{1}{\nu^2}\left(\Omega_3^2(\beta_2) + \frac{\partial \Omega_3^2(\beta_2)}{\partial \beta_2}\epsilon \sin t + \cdots\right)\xi \\ &\sim -\left(\frac{\Omega_3}{\nu}\right)^2\left(1 + \frac{2}{\Omega_3}\frac{\partial \Omega_3}{\partial \beta_2}\epsilon \sin t\right)\xi \\ &= -\omega'^2(1 + \epsilon' \sin t)\xi.\end{aligned} \tag{4}$$

The amplitude of the parametric excitation $\epsilon'$ is proportional to $\frac{2}{\Omega_3}\frac{\partial \Omega_3}{\partial \beta_2}$. Thus, parametric excitation is more effective when $\Omega_3$ becomes small and steep, *i.e.* near the bifurcation point.

In the Mathieu equation, the stable equilibrium point $\xi = 0$ loses its stability if $\omega'$ is close to or equal to an integer and half-integer and $\epsilon'$ is greater than zero. In this situation, a standing ILM will become unstable and may become mobile. To investigate the region of ILM stability loss, the ILM position can be defined using the lattice energy distribution [19]:

$$\begin{aligned}h = \sum_{n=1}^{N} \Bigg\{ &\left(\frac{1}{2}\dot{u}_n^2 + \frac{1}{2}u_n^2 + \frac{\beta_1}{4}u_n^4\right) e^{i\frac{2\pi}{N}n} \\ &+ \left(\frac{\alpha}{2}(u_n - u_{n-1})^2 + \frac{\beta_2}{4}(u_n - u_{n-1})^4\right) e^{i\frac{2\pi}{N}(n+\frac{1}{2})} \Bigg\},\end{aligned} \tag{5}$$