



Uncertainty relations for approximation and estimation



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ABSTRACT

We present a versatile inequality of uncertainty relations which are useful when one approximates an observable and/or estimates a physical parameter based on the measurement of another observable. It is shown that the optimal choice for proxy functions used for the approximation is given by Aharonov's weak value, which also determines the classical Fisher information in parameter estimation, turning our inequality into the genuine Cramér–Rao inequality. Since the standard form of the uncertainty relation arises as a special case of our inequality, and since the parameter estimation is available as well, our inequality can treat both the position–momentum and the time–energy relations in one framework albeit handled differently.

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1. Introduction

Uncertainty relations lie undoubtedly at the heart of quantum mechanics, characterizing the indeterministic nature of microscopic phenomena which stem from the incompatibility of simultaneous measurement of two non-commuting observables, as typically exemplified by position and momentum. Soon after the celebrated exposition of Heisenberg's tradeoff relation between error and disturbance [1], there appeared a revised form called the Robertson–Kennard (RK) inequality [2,3] which refers to the relation in standard deviation in independently performed measurements on the two observables. Because of its mathematical clarity and universal validity, the latter has now become a standard textbook material.

Later, these relations were elaborated from operational viewpoints by taking account of the measurement device, and this has yielded, e.g., the Arthurs–Goodman inequality [4] and the Ozawa inequality [5] which concern a mixed relation among error, disturbance and standard deviation. Apart from these, the uncertainty relations on error and disturbance have also been analyzed in quantum estimation theory [6].

On the other hand, the uncertainty relation between time and energy has to be dealt with quite independently from these, due to the lack of a genuine time operator conjugate to the Hamiltonian. For this, several ingenious frameworks have been proposed, including the one devised by Mandelshtam–Tamm [7] and that by

Helstrom [8], where the uncertainty relation is shown to be identified with a quantum version of the Cramér–Rao inequality [9] in estimation theory.

In this paper, we present a novel inequality of uncertainty relations for analyzing the error of approximating an observable based on the measurement of another observable through an appropriate choice of proxy functions. Since the standard deviation may be regarded as a special case of our approximation error, our inequality can formally be considered as an extension of the RK inequality. Moreover, instead of approximating an observable, we may also choose to estimate a physical parameter pertinent to the observable, so that the time–energy relation can be treated along with the position–momentum relation.

Interestingly, in both approximation and estimation, Aharonov's weak value [10] of the concerned observable arises as a key geometric ingredient, deciding the optimal choice for the proxy functions. We shall also find in the context of parameter estimation that the weak value determines the classical Fisher information and turns our inequality into the Cramér–Rao inequality.

2. Uncertainty relation for approximation

Before presenting our uncertainty relation, let us recall the most familiar form of the relations, i.e., the Robertson–Kennard (RK) inequality,

$$\|A - \langle A \rangle\| \cdot \|B - \langle B \rangle\| \geq \frac{1}{2} |\langle [A, B] \rangle|, \quad (1)$$

valid for two observables A, B . Here, $\langle A \rangle = \langle \psi | A | \psi \rangle$ is the expectation value of A under a given (normalized) state $|\psi\rangle$, $[A, B] =$

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$AB - BA$ is the commutator, and $\|X\| = \sqrt{\langle X^2 \rangle}$ is the operator seminorm defined for a self-adjoint operator X . The RK inequality gives the lower bound for the product of the standard deviation $\sigma(X) = \sqrt{\text{Var}[X]} = \|X - \langle X \rangle\|$ for the two observables A and B . Needless to say, the lower bound of the RK inequality (1) takes the state-independent value $\hbar/2$ when A and B are canonically conjugate to each other, $[A, B] = i\hbar$.

In place of the standard symmetric treatment of the observables in the RK inequality (1), we now consider an asymmetric but more versatile form given by

$$\|A - f(B)\| \cdot \|g(B)\| \geq \frac{1}{2} |\langle [A, g(B)] \rangle|, \quad (2)$$

valid for arbitrary self-adjoint operator functions $f(B)$ and $g(B)$ of B (the advantage of this asymmetric treatment will become clear shortly). More explicitly, the operator $f(B)$ is defined from a function $f(b)$ of b through the spectral decomposition,

$$f(B) = \int f(b) |b\rangle \langle b| db, \quad (3)$$

where $\{|b\rangle\}$ is the basis set of the eigenstates $B|b\rangle = b|b\rangle$, and the integral in (3) is understood to imply summation when the eigenvalues are discrete. The operator $g(B)$ can be defined analogously from a function $g(b)$. In our discussion, $f(b)$ and $g(b)$ are assumed real so that both $f(B)$ and $g(B)$ are self-adjoint, but the definition (3) can be applied to any normal operators using complex functions. The RK inequality (1) arises from (2) as a special case by letting $f(B) = \langle A \rangle$ (realized by the constant function $f(b) = \langle A \rangle$) and $g(B) = B - \langle B \rangle$.

The proof of the inequality (2) goes precisely the same way as that of the RK inequality. Namely, given two self-adjoint operators X, Y , we have $\|X\|^2 \cdot \|Y\|^2 \geq |\langle XY \rangle|^2$ by the Cauchy-Schwarz (CS) inequality. We also have $|\langle XY \rangle|^2 = |\langle [X, Y]/2 + \{X, Y\}/2 \rangle|^2$ where $\{X, Y\} = XY + YX$, but since $\langle [X, Y]/2 \rangle$ is purely imaginary whereas $\langle \{X, Y\}/2 \rangle$ is real, we obtain $|\langle XY \rangle|^2 = |\langle [X, Y]/2 \rangle|^2 + |\langle \{X, Y\}/2 \rangle|^2 \geq |\langle [X, Y]/2 \rangle|^2$. Combining these, and taking the square root of the two sides, we arrive at $\|X\| \cdot \|Y\| \geq \frac{1}{2} |\langle [X, Y] \rangle|$. Since X and Y are arbitrary, we may put $X = A - f(B)$ and $Y = g(B)$ to obtain our inequality (2).

Although our inequality is merely an asymmetric generalization of the RK inequality, the acquired form (2) allows for a novel viewpoint on the uncertainty relation. Specifically, noting that $\|A - f(B)\|$ gives a measure for the ‘distance’ between the two observables A and $f(B)$, we may regard (2) as an inequality giving the lower bound for the distance under the choice of $f(B)$ and $g(B)$. This will be made more apparent by introducing $\bar{g}(B) = g(B)/\|g(B)\|$ to rewrite (2) as

$$\min_f \|A - f(B)\| \geq \max_{\bar{g}} \frac{1}{2} |\langle [A, \bar{g}(B)] \rangle|. \quad (4)$$

This indicates that the minimal distance between A and the family of all self-adjoint operators $f(B)$ generated by B , or the minimal error in the approximation of A from the measurement of B in terms of real proxy functions $f(b)$, is dictated by the maximal degree of non-commutativity of A with respect to the family of all self-adjoint operators $\bar{g}(B)$ normalized as $\|\bar{g}(B)\| = 1$.

Clearly, our inequality will be useful in the operational context in which one measures only B and approximates A out of the measurement result by choosing the proxy function $f(B)$ properly. In this context, the choice $f(B) = \langle A \rangle$, which makes the distance into the standard deviation $\sigma(A)$, is far from the optimal one, having only its expectation value right. In fact, we shall see shortly that the optimal choice for $f(b)$ is provided explicitly by the real part of the weak value $A_w(b)$ which is defined in the quantum process specified by the initial state $|\psi\rangle$ and the final state $|b\rangle$

(see (5)). Under this optimal choice together with $g(B) = B - \langle B \rangle$, our inequality (2) yields an uncertainty relation stricter than the RK inequality. The freedom of choice for $g(b)$ may further be exploited for considering parameter estimation, that is, for estimating a parameter t that specifies the state, as exemplified later by the situation in which the state varies unitarily with the generator A . In this context, the optimal choice for $g(b)$ turns out to be given by the imaginary part of the weak value $A_w(b)$.

3. Optimal choice and the weak value

In what follows, we assume for simplicity the non-degeneracy of B and the condition $\langle b|\psi\rangle \neq 0$ for all $|b\rangle$, which can always be ensured if one chooses B appropriately with respect to the given state $|\psi\rangle$. The primary role of this restriction is to avoid the mathematical elaborations required to introduce the weak value

$$A_w(b) = \frac{\langle b|A|\psi\rangle}{\langle b|\psi\rangle} \quad (5)$$

in a completely general manner, so that we may regard it simply as a function $b \mapsto A_w(b)$ from the eigenvalues of the self-adjoint operator B to complex numbers. (In order to lift this restriction, we need to adopt the definition of the weak value being an equivalence class of the family of functions with ambiguity at the singular points $\langle b|\psi\rangle = 0$, for which our argument below goes through without an essential change.) Here, note also the state-dependence of the weak value $A_w(b)$.

Now, in order to see the statements made above, let us first note the identity,

$$\begin{aligned} A|\psi\rangle &= \int |b\rangle \langle b|A|\psi\rangle db \\ &= \int A_w(b) |b\rangle \langle b|\psi\rangle db = A_w(B)|\psi\rangle, \end{aligned} \quad (6)$$

where

$$A_w(B) = \int A_w(b) |b\rangle \langle b| db \quad (7)$$

is the operator function defined analogously to (3) with $f(b)$ replaced formally by the weak value $A_w(b)$ given in (5). Note that, due to the complex nature of $A_w(b)$, the operator $A_w(B)$ is not necessarily self-adjoint but rather normal, and that it is dependent on the choice of the state $|\psi\rangle$.

Then, from (6) we have

$$\langle f(B)A \rangle = \langle f(B)A_w(B) \rangle \quad (8)$$

for any self-adjoint $f(B)$, and thus

$$\begin{aligned} \text{Re} \langle f(B)A \rangle &= \langle f(B)\text{Re}A_w(B) \rangle, \\ \text{Im} \langle f(B)A \rangle &= \langle f(B)\text{Im}A_w(B) \rangle, \end{aligned} \quad (9)$$

which specifically leads to

$$\begin{aligned} \langle A \rangle &= \langle \text{Re}A_w(B) \rangle, \\ 0 &= \langle \text{Im}A_w(B) \rangle \end{aligned} \quad (10)$$

for the choice of the constant function $f(b) = 1$. Another consequence of (6) is

$$\|A\|^2 = \|\text{Re}A_w(B)\|^2 + \|\text{Im}A_w(B)\|^2. \quad (11)$$

These statistical properties on average and correlation suggest that the operator $\text{Re}A_w(B)$ may furnish the optimal proxy function for A minimizing the distance $\|A - f(B)\|$. That this is indeed the case can be confirmed at once from the ‘Pythagorean identity’

$$\|A - f(B)\|^2 = \|A - \text{Re}A_w(B)\|^2 + \|\text{Re}A_w(B) - f(B)\|^2 \quad (12)$$

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