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On bounded and unbounded dynamics of the Hamiltonian system for unified scalar field cosmology

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ABSTRACT

This paper is devoted to the research of global dynamics for the Hamiltonian system formed by the unified scalar field cosmology. We prove that this system possesses only unbounded dynamics in the space of negative curvature. It is found the invariant domain filled only by unbounded dynamics for the space with positive curvature. Further, we construct a set of polytopes depending on the Hamiltonian level surface that contain all compact invariant sets. Besides, one invariant two dimensional plane is described. Finally, we establish nonchaoticity of dynamics in one special case.

1. Introduction

In this paper we examine the Hamiltonian system which describes dynamics of the scalar field Friedman–Lemaitre–Robertson–Walker cosmological model within the framework of the unified dark matter scenario, see [3,4]. This system is defined by the Hamiltonian

$$H_\sigma = \frac{1}{2}(p_x^2 - \sigma\omega_1^2 x^2 - p_y^2 + \omega_2^2 y^2) - \frac{3^{4/3}k}{2}(y^2 - x^2)^{1/3}, \sigma = \pm 1. \quad (1)$$

This Hamiltonian presented by Lukes-Gerakopoulos et al. in papers [2,20] was derived from the formula for the potential

$$U(\phi) = c_1 \cosh^2(c\phi) + c_2$$

given in [3], see also [9,10]. Here ϕ is the scalar field and constants c; c_1; c_2 are real numbers. The dynamical system corresponded to (1) is described by equations

$$\dot{x} = p_x \quad (2)$$
$$\dot{y} = -p_y$$
$$\dot{p}_x = (\sigma\omega_1^2 - \frac{3^{1/3}k}{(y^2 - x^2)^{2/3}})x$$
$$\dot{p}_y = (-\omega_2^2 + \frac{3^{1/3}k}{(y^2 - x^2)^{2/3}})y$$

which are well-defined on the set $M := \{y > 0; y > |x|\} \subset \mathbf{R}^4$ and loose their sense on the boundary of M. In these equations the parameter $k \neq 0$ is used for characterizing Gaussian curvature of the space, i.e. the nonflat case is considered; ω_i^2, $i = 1, 2$, are oscillators' "frequencies", see [2,20] for details. The parameter $\sigma = \pm 1$ is introduced for a brief presentation of our results. Here and below points of the space $\mathbf{R}^4$ are given by vectors $(x, y, p_x, p_y)^T$.

We recall that authors of [20] studied numerically the system (2) with $H_\sigma = 0$ and values of the curvature k closed to zero in order to detect chaos by exploiting properties of the deviation vector. Besides, dynamics of (2) in the flat case $k = 0$ was analyzed in [2] by computing main cosmological functions and cosmological constraints.

In this work we explore dynamics of (2) for various values of k and level sets $H_\sigma^{-1}(l)$, with $k; l \in R$. Our main goal is twofold. Firstly, we describe invariant domains filled only by unbounded dynamics. This problem is stated as a problem of finding level sets $H_\sigma^{-1}(l)$ which are free of compact invariant sets. Secondly, we compute one l-parametric set of polytopes $\Theta(l)$ containing all compact invariant sets located in $H_\sigma^{-1}(l)$. It means that all bounded motions of the system (2) are presented only in $\Theta(l)$ and if some positive (negative) half trajectory in $H_\sigma^{-1}(l)$ is bounded then its ω (α)-limit

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set is contained in $\Theta(l)$. Here we recall that the role of α- and ω-limit sets is discussed in many works in the context of behavior of cosmological models at early and late times, see e.g. [11,27] and others.

Finding invariant domains which are free of compact invariant sets is of substantial interest for the analysis of unbounded dynamics. This problem is nontrivial for Hamiltonian systems because these systems preserve phase volumes and, as a result, it is difficult to derive conclusions on the existence of unbounded domains filled by unbounded dynamics only. It is worth to mention that unbounded dynamics studies of nonlinear systems draw attention of many researchers during the last decades, see e.g. works concerning unbounded dynamics of Rössler system; Rikitake system; robust unbounded attractors; open integrable Hamiltonian systems; in the context of proofs of nondissipativity in the sense of Levinson for the phase flow, using Poincaré compactification, analysis of trajectories with finite-time blow-ups and other topics, see e.g. [1,5–8,12,18]. For Hamiltonian systems with more than one degree of freedom their dynamics may vary significantly for different level surfaces. For example, we can meet a situation when there are no compact invariant sets in some level surfaces while in others they may exist. Therefore one may localize parameters defining Hamiltonian level surfaces which do not contain any compact invariant sets. This problem is partially solved in our paper.

Another important problem explored in this work concerns the localization of ω-limit sets for the Hamiltonian system (2). The interest to this problem is caused the fact that ω-limit sets consist of points characterizing the late time states of the universe. Therefore their localization domains can be considered as a space approximation of ω-limit sets and may be used for a prediction of possible late time dynamics. This comment is also true in a relation to a localization of α-limit sets as a possible tool in studies of early time state of the universe.

Our approach has been recently exploited in papers [25,26] for analysis of unbounded dynamics of various cosmological systems and is based on the solution of the localization problem of compact invariant sets, [15,16], and the LaSalle theorem. Besides, we mention that papers [17,22–24] contain results on analysis of global dynamics for various cosmological systems which is based on the localization method of compact invariant sets as well.

The phase flow of a Hamiltonian system preserves phase volumes in virtue of the Liouville's theorem. As a consequence, we have that any Hamiltonian system does not possess an attractor, i.e. a compact invariant set which attracts all points from some set with a non-empty interior. Objects possessing some features of attractors have appeared in a research of various Hamiltonian systems formed by cosmological scalar fields. They are called apparent attractors in the opposite to true attractors mentioned above. We recall that a detailed discussion concerning true attractors and apparent attractors is contained in [21]. In the paper [20] the numerical study of the apparent attractor is realized for the space with $k = 10^{-3}$.

The localization method of compact invariant sets described briefly in Section 2 is capable to localize apparent attractors because this localization method is based upon solving the conditional extremum problem rather than Lyapunov ideas, see discussion concerning a comparison of these methods in the end of [16]. Our approach applied to the system (2) provides us compact localization domains in the form of one-parameter set of polytopes $\Theta(l)$.

The structure of the paper is as follows. In Section 2 we introduce helpful preliminaries. In Sections 3–5 we assume that $\omega_2^2 > \omega_1^2$. In Section 3 we describe the cases for which the system (2) has no compact invariant sets: 1) in the invariant domain M for the case of the open universe ($k < 0$), $\sigma = \pm 1$, and 2) in sets $M \cap H_\sigma^{-1}(l)$, with $l \in (-\infty, l_{\min}(\sigma))$, for the case of the closed universe ($k > 0$) and $\sigma = \pm 1$. Here the value $l_{\min}(\sigma)$ is defined precisely by the condition that the level set $H_\sigma^{-1}(l_{\min}(\sigma))$ contains the unique equilibrium point $E = (0, 3^{1/4}k^{3/4}\omega_2^{-3/2}, 0, 0)$. The cosmological meaning of these assertions consists in the observation that the universe cannot recollapse and its expansion continues eternally. Further, Section 4 contains main results of this work which are derived for the case of the closed universe. Namely, we construct one l-parameter set of polytopes containing all compact invariant sets in invariant sets $M \cap H_\sigma^{-1}(l)$, with $l \geq l_{\min}(\sigma)$. These polytopes are obtained in the 4-step procedure which is based on finding bounds for real roots of some cubic equations. As a result, we establish that the recollapse of the universe may occur only within these polytopes. In Section 5 we prove that the system (2) possesses the invariant plane $\Pi := \{x = p_x = 0\}$. In Section 6 we establish a curious fact that in case $\sigma = 1$ and $\omega_1^2 > \omega_2^2$ the system (2) has a nonchaotic dynamics. Besides, the plane Π contains the ω-limit set of each bounded half trajectory in M. Concluding remarks are presented in Section 7.

2. Useful background and preliminaries

Let us introduce a system

$$\dot{x} = F(x) \tag{3}$$

where $F(x)$ is a n-variate rational vector function on $\mathbf{R}^n$ which is well-defined on the open domain $W \subset \mathbf{R}^n$. By $\varphi(x,t)$ we denote a solution of (3), with $\varphi(x,0) = x$ for any $x \in W$. Let $h(x)$ be a differentiable function on the domain $U \subseteq W$ such that h is not the first integral of (3). By $L_F h$ we denote the Lie derivative of the function h and by $S(h;U)$ we denote the set $\{x \in U \mid L_F h(x) = 0\}$. Further, we define $h_{\inf}(U) := \inf\{h(x) \mid x \in S(h;U)\}$; $h_{\sup}(U) := \sup\{h(x) \mid x \in S(h;U)\}$. The following result, [15,16], is used in this paper:

Assertion 1. *1) Assume that $S(h;U) \neq \emptyset$. Then each compact invariant set Γ of (3) contained in U is located in the set $K(U) := \{h_{\inf}(U) \leq h(x) \leq h_{\sup}(U)\} \cap U$ as well; $K(U)$ is called a localization set. 2) If $S(h;U) = \emptyset$ then the system (3) has no compact invariant sets contained in U.*

Also, we shall exploit the following result, see [19].

Assertion 2. *Consider cubic equation $u^3 + qu + r = 0$, $r \neq 0$. 1) If this equation has only real roots β_i, $i = 1,2,3$, then $\max \beta_i^2 \leq -4q/3$. 2) If this equation has the unique real root β then $\beta^2 > -4q/3$.*

In order to get the upper bound for β^2 in the second case we apply the change $u = v^{-1}$, with $v \neq 0$, in this cubic equation and come to the equation $v^3 + qr^{-1}v^2 + r^{-1} = 0$ which also has the unique real root equal to β^{-1}. Then we eliminate the quadratic term in the last equation by another change of variables and apply Assertion 2. As a result, we come to the estimate $\beta^2 < 9r^2q^{-2}/4$.

In the sequel we shall exploit the notation $\psi(x,y) = \frac{3^{1/3}k}{(y^2 - x^2)^{2/3}}$.

3. On unbounded dynamics of the system (2)

Below by f we denote the corresponding vector field of (2) and we shall use the notation $S(h) := S(h; M)$. In this section we describe domains filled only by unbounded dynamics. Firstly, we mention that if $k > 0$ then the system (2) has the unique equilibrium point E located in M; if $k < 0$ then the system (2) has no equilibrium points located in M, [20]. The authors of [20] concluded from this observation that in case of the negative curvature the system (2) has only unbounded dynamics and the universe