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Approximate solutions to the quantum problem of two opposite charges in a constant magnetic field

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ABSTRACT

We consider two particles of equal mass and opposite charge in a plane subject to a perpendicular constant magnetic field. This system is integrable but not superintegrable. From the quantum point of view, the solution is given by two fourth degree Hill differential equations which involve the energy as well as a second constant of motion. There are two solvable approximations in relation to the value of a parameter. Starting from each of these approximations, a consistent perturbation theory can be applied to get approximate values of the energy levels and of the second constant of motion.

1. Introduction

In a previous paper [1] we have considered a kind of Landau system with two charged particles in a plane. The charges have the same absolute value and opposite sign and the same mass. These particles are subject to a constant perpendicular magnetic field. The same situation, under a different point of view has been discussed in some recent papers [2,3].

This model has considerable interest in Physics. For instance, it can be interpreted as a positronium system, or as a Frenkel or Mott–Wannier exciton [4]. Other applications have been studied in [5,6]. The model is also closely related to the system of a particle under two fixed gravity centers, a classical subject [7].

We have shown in [1] that this system may be studied from the point of view of either classical or quantum mechanics. The transition from the former to the latter is achieved through canonical quantization [8]. This system has four independent commuting constants of motion, or symmetries. Classically, the commutation is defined in terms of Poisson brackets. This system is integrable although not superintegrable.

In the classical analysis presented in [1], we have used two of these constants, written in compact form as the components of the two dimensional vector $\boldsymbol{\mu}$, in order to reduce by two the number of degrees of freedom, so that we have an effective two dimensional system.

The resulting Hamiltonian is a sum of a kinetic term plus an effective potential, which is given by the sum of a Coulomb potential plus a shifted harmonic oscillator. Along with this effective Hamiltonian, we have an additional constant of motion, denoted by T. This fact allows to separate the system in elliptic coordinates.

In the present Letter, we focus our interest on the quantum version of this model. Within this quantum context, the separation in elliptic coordinates of the effective system leads to a pair of equations. One is a fourth degree periodic Hill equation, while the second one is a similar modified Hill equation with hyperbolic functions [9,10]. Up to our knowledge, analytic solutions for these equations are not known.

Along this presentation, we shall discuss the possibility of obtaining approximate solutions of these equations by means of a procedure based on perturbation theory. In our calculations, we shall use $\mu := |\boldsymbol{\mu}|$ as natural perturbative parameter. This $\boldsymbol{\mu}$, to be defined in the next section (right after (6)), is a constant of motion and gives the position of the center of the displaced harmonic oscillator.

The zero order of perturbation will approximately describe the system either for $\mu << 1$ or for $\mu >> 1$. In the first case, the Coulomb term will be dominant with respect to the oscillatory term, now used as a perturbation. The situation is reversed in the

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second case, where the oscillatory term is dominant and the perturbation is given by the Coulomb part.

We shall see that in the zero order approximation valid for $\mu << 1$, or Coulomb approximation, both trigonometric and hyperbolic Hill equations become a pair of equations of the type discussed by Razavy in [11,12], which are solvable. These equations were known by some authors as the hyperbolic Wittaker–Hill equations. Nevertheless, since we are using as the reference the work by Razavy and following the use of some recent authors, we prefer to use the terminology *Razavy type* equations or simply, *Razavy* equations. Their solutions coincide with the bound solutions of the Coulomb problem and take significant values close to the origin of the potential. This means that the relative position between both particles keeps very small.

When $\mu >> 1$, we can again approximate both Hill equations by Razavy type equations. Their solutions describe the solutions for the harmonic oscillator in elliptic coordinates. These solutions are the bound states of the harmonic oscillator and correspond to much larger values of μ.

Using these zero order approximations as the unperturbed systems, we propose perturbations of first order. In this approach, we assume that the representation in terms of elliptic coordinates is valid for any order of perturbation.

This Letter is organized as follows: In Section 2, in order to orient the reader and for the sake of completeness, we summarize the results obtained in [1] with some additional information. We give in Section 3 the exact resolution of the zero order Razavy equations, where we pay an special attention to the correct choice of the boundary conditions and its consequences. In Section 4, we discuss the first order perturbative approach to solutions. Explicit expressions are left to Supplementary Material. We close our discussion with some Concluding Remarks.

2. Presentation of the problem

In this section, we briefly review the treatment given in [1]. Let us begin with the classical description of the model. The Hamiltonian describing two charged particles, with charges e and $-e$, of equal mass m, interacting among themselves by the Coulomb potential and subject to an external constant magnetic field perpendicular to the plane in which the particles move is given by $(c = 1)$:

$$H = \frac{1}{2m}[(\mathbf{p}^{(1)} - eA(\mathbf{x}^{(1)}))^2 + (\mathbf{p}^{(2)} + eA(\mathbf{x}^{(2)}))^2] - \frac{e^2}{|\mathbf{x}^{(1)} - \mathbf{x}^{(2)}|} . \tag{1}$$

Here by $\mathbf{x}^{(k)}$, $\mathbf{p}^{(k)}$, $k = 1, 2$, we denote positions and linear momenta of both particles. The vector potential $\mathbf{A}(\mathbf{x})$ is taken in the symmetric gauge,

$$A_i(\mathbf{x}) = h\,\varepsilon_{ij}x_j , \qquad \mathbf{A}(\mathbf{x}) = (h\,x_2, -h\,x_1) , \tag{2}$$

where ε_{ij} is the totally antisymmetric tensor in two dimensions. We are using the convention of summation over repeated indices. The magnetic field is parallel to the z axis with intensity $B = -2h$. For each particle k $(k = 1, 2)$, we define a kinematic momentum $\boldsymbol{\pi}^{(k)}$ with components,

$$\begin{aligned}\pi_i^{(1)} &= p_i^{(1)} - eA_i^{(1)} = p_i^{(1)} - e\varepsilon_{ij}x_j^{(1)}h ,\\ \pi_i^{(2)} &= p_i^{(2)} + eA_i^{(2)} = p_i^{(2)} + e\varepsilon_{ij}x_j^{(2)}h .\end{aligned} \tag{3}$$

Next, define the components, Π_i of the *total momentum* and the center of mass (c.o.m.) coordinates Q_i, $i = 1, 2$, by

$$\begin{aligned}\Pi_i &:= \pi_i^{(1)} + 2eh\varepsilon_{ij}x_j^{(1)} + \pi_i^{(2)} - 2eh\varepsilon_{ij}x_j^{(2)} ,\\ Q_i &:= \frac{1}{2}(x_i^{(1)} + x_i^{(2)})\end{aligned} \tag{4}$$

and the relative momentum and coordinates as

$$\pi_i := \frac{1}{2}(\pi_i^{(1)} - \pi_i^{(2)}) , \qquad q_i := x_i^{(1)} - x_i^{(2)} , \qquad i = 1, 2 . \tag{5}$$

Due to the fact that the total charge of the system vanishes, the functions $\{\mathbf{q}, \boldsymbol{\pi}, \mathbf{Q}, \boldsymbol{\Pi}\}$ constitute a canonical coordinate set. We could have equally discussed the case of two particles with different mass m_1, m_2; then, a similar canonical set would have been obtained.

In terms of these new coordinates, the initial Hamiltonian (1) has the following form:

$$\begin{aligned}H &= \frac{1}{4m}\boldsymbol{\Pi}^2 - \frac{eh}{m}\,\varepsilon_{ij}\Pi_i q_j + \frac{1}{m}\,\boldsymbol{\pi}^2 + \frac{e^2h^2}{m}\mathbf{q}^2 - \frac{e^2}{q}\\ &= \frac{1}{m}\,\boldsymbol{\pi}^2 + \frac{e^2h^2}{m}\,(\mathbf{q} + \frac{\boldsymbol{\mu}}{2})^2 - \frac{e^2}{q} ,\end{aligned} \tag{6}$$

with $q := |\mathbf{q}|$, $\mu_j = -\varepsilon_{ij}\Pi_i/eh$ and $\mu := |\boldsymbol{\mu}|$. As the coordinates $\mathbf{Q}$ are cyclic, the components of the "total momentum" $\boldsymbol{\Pi}$ are constants of motion given in terms of $\boldsymbol{\mu}$. From (6), we conclude that the effective system consists of a particle, with a reduced mass $m/2$ and charge e, in the plane under the influence of a Coulomb potential set at the origin with charge $-e$, plus a shifted harmonic oscillator potential with angular frequency $\omega = 2eh/m = e|B|/m$, which is the cyclotron frequency, being $|B| = 2h$ the magnetic field intensity.

As is clear from (4), the constant of motion $\boldsymbol{\Pi}$ is the sum of the generators of magnetic translations for each particle, just as they are defined for the Landau system of a single particle in a constant magnetic field. In the Landau system, the values of $\boldsymbol{\Pi}$ give the center of the circular trajectories; in this case, the values of $\boldsymbol{\Pi}$ determine the relative position of the two centers of the Coulomb and oscillator effective potentials.

As the first term, $\frac{1}{4m}\boldsymbol{\Pi}^2 = \frac{(eh\mu)^2}{4m}$, in (6) is a constant, it will be hereafter dropped to simplify the expressions. Nevertheless, it will be recovered later in order to interpret the approximation $\mu >> 1$.

As shown in [1], this system has two independent constants of motion:

$$H = \frac{\boldsymbol{\pi}^2}{m} + U(\mathbf{q}) ; \qquad T := \pi_i\, g_{ij}(\mathbf{q})\pi_j + \Phi(\mathbf{q}) . \tag{7}$$

Here, H is the effective Hamiltonian given in (6), without the above mentioned constant term. The second constant of motion T includes a "kinetic term" given by[1]:

$$\begin{aligned}&\pi_i g_{ij}(\mathbf{q})\pi_j = L^2 + (\mu_1\pi_2 - \mu_2\pi_1)L = \frac{1}{2}(L\cdot L' + L'\cdot L) ,\\ &L := q_1\pi_2 - q_2\pi_1 ,\end{aligned} \tag{8}$$

where L and L' are the angular momenta with respect to the origin and to the point $-\boldsymbol{\mu}$, respectively. By the way, this term has been already obtained by Erikson–Hill in [14] for the two center problem. The kinetic tensor can also be expressed as

$$g_{ij} = q_i^* q_j^* + \frac{1}{2}(\mu_i^* q_j^* + q_i^* \mu_j^*) , \tag{9}$$

where, $q_i^* = \varepsilon_{ik}q_k$, $\mu_i^* = \varepsilon_{ik}\mu_k$. The "potential term" Φ is given by

$$\Phi(\mathbf{q}) = \frac{2m\,e^2}{\mu^2}\,\frac{\mathbf{q}\cdot\boldsymbol{\mu}}{q} + \frac{e^2h^2}{4}\,(q^2\mu^2 - (\mathbf{q}\cdot\boldsymbol{\mu})^2) . \tag{10}$$

Note that T is a constant of motion in the sense that $\{H, T\} = 0$, where $\{\cdot, \cdot\}$ stands for Poisson bracket. Then with the help of T, we can separate the system using the confocal elliptic coordinates

[1] Note that in the quantum case one has to use a symmetrized expression.