Physics Letters A 380 (2016) 1839–1847

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Inner structure of vehicular ensembles and random matrix theory

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ARTICLE INFO

Article history:
Received 23 October 2015
Received in revised form 23 March 2016
Accepted 24 March 2016
Available online 29 March 2016
Communicated by C.R. Doering

Keywords:
Vehicular traffic
Time-clearance distribution
Random matrix theory

ABSTRACT

We introduce a special class of random matrices (DUE) whose spectral statistics corresponds to statistics of microscopical quantities detected in vehicular flows. Comparing the level spacing distribution (for ordered eigenvalues in unfolded spectra of DUE matrices) with the time-clearance distribution extracted from various areas of the flux-density diagram (evaluated from original traffic data measured on Czech expressways with high occupancies) we demonstrate that the set of classical systems showing an universality associated with Random Matrix Ensembles can be extended by traffic systems.

Introduction

Connection between Random Matrix Ensembles and certain transport systems is not new (see e.g. [1] for a general overview). Indeed, in Refs. [2,3] authors drew the attention to the bus transportation system in Cuernavaca, Mexico, where a peculiar transfer of information inside the system led, surprisingly, to the universal configuration of gaps among buses. In more detail: Since the complicated traffic conditions in Mexico work as an effective randomizer the time headways among buses (if not influenced by any additional information) obey the exponential distribution (bus clustering). This is in conflict with the economic effort of drivers to maximize the number of transported passengers. In order to avoid such an unpleasant clustering effect the bus drivers engage people who record the arrival times of buses at significant places. Arriving at a checkpoint, the driver receives the information of when the previous bus passed that place. Knowing the time interval the driver optimizes the distance to the preceding bus by either slowing down or speeding up. In such a way the obtained information leads to a vicarious interaction between buses and changes the statistical properties of the gap distribution. The rigorous study [4] confirmed that the detected link between Cuernavaca buses and Gaussian unitary ensemble (GUE) is not accidental. Authors of the research formulated a relevant microscopic model-scheme and analytically proved that the time headway distribution of buses conforms to the level spacing distribution for GUE. This surprising knowledge attracted an attention of scientists (e.g. [1]) since systems connected to Random Matrix Theory are universal (in a certain sense).

Until now, natural endeavor to interconnect Random Matrix Theory (RMT) with vehicular systems has not led to a success. However, a partial progress has been achieved in [5–7,9,10] where proved that microscopical arrangement of vehicles can be predicted by means of a certain one-dimensional gas inspired by the Dyson's gases that are well-known in RMT. Moreover, another attempt to describe (analytically) a microstructure of vehicular ensembles with help of statistical instruments (e.g. [11–13]) led us to believe that our effort may be successful. Therefore, the main goal of this work is to find (and analyze) a new class of random matrices whose spectral properties correspond to a micro-structure of real-road traffic samples.

This paper is organized as follows. In the first section we introduce and analyze the class of the so-called damped matrices (DUE). We are focused on the level density, the procedure of unfolding, and the level spacing distribution. Connection to the theory of one-dimensional traffic gases is discussed in the second section. Section 3 brings in a comparison between the level spacing distribution of DUE matrices and the time-clearance distributions extracted from expressway data samples. This is followed by corresponding discussion and conclusions in the final section. Subsidiary derivations and additional statistical tests are included in the mathematical Appendix.

1. DUE – damped unitary ensembles

Being inspired by the work [14] (studying Calogero–Moser models with various potentials) we introduce the g-parameterized class $\mathrm{DUE}_g(N)$ whose $N \times N$ matrices $H_g = (h_{kj}(g))_{k,j=1}^{N}$ fulfill the following axioms:

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http://dx.doi.org/10.1016/j.physleta.2016.03.037
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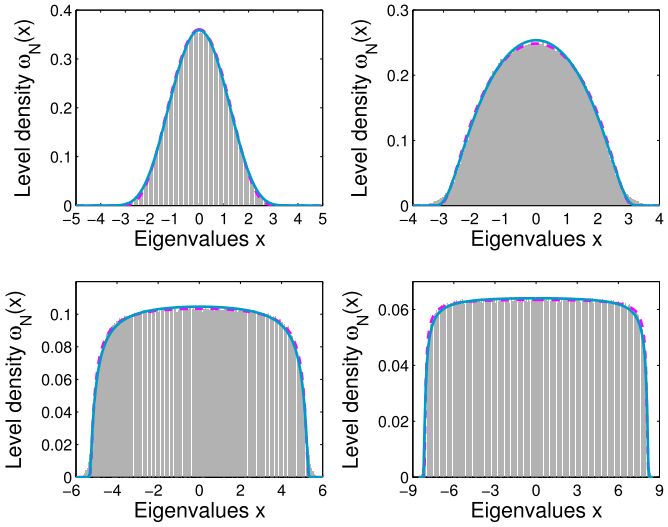

Fig. 1. Graphs of the level density $\omega_N(x)$. Histograms have been determined for five thousand realizations of DUE(256) matrices with parameters $g = 0.2$ (a northwest sub-figure), $g = 0.5$ (a northeast sub-figure), $g = 1.5$ (a southwest sub-figure), and $g = 2.5$ (a southeast sub-figure). Solid curve (blue)/dashed curve (magenta) represent the approximation (3) for values ε, ϑ estimated with help of CSC/KE methods, respectively. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

(i) the parameter $g \geqslant 0$ is fixed;
(ii) elements $h_{kk}(g)$ are chosen as independent Gaussian variables with zero mean and unit variance, i.e. $h_{11}(g), h_{22}(g), \ldots, h_{NN}(g) \backsim \mathrm{N}(0,1)$ are i.i.d.;
(iii) if $k \neq j$ then

$$h_{kj}(g) = \frac{2\pi\,\mathrm{i}\,g}{N \sinh[(2\pi\,(k-j)/N]}, \tag{1}$$

i.e. the off-diagonal elements are deterministic.

Owing to the definition the off-diagonal elements are purely imaginary complex numbers whose absolute value become smaller with distance from the diagonal. Therefore we refer such a class to as the Damped Unitary Ensembles. In the context of the article [14] the matrices in $\mathrm{DUE}_g(N)$ represent simplified versions of the $N \times N$ Lax matrices derived for hyperbolic Calogero–Moser model (see [14] for details).

1.1. Level density for DUE matrices

Let $\sigma(H_g)$ be the spectrum of H_g, i.e. $\sigma(H_g) = \{x_{(k)} \in \mathbf{C} : k = 1, 2, \ldots, N\}$ is a set of ordered eigenvalues $x_{(1)} \leqslant x_{(2)} \leqslant \ldots \leqslant x_{(N)}$. Since $H_g = H_g^{\sharp}$, where $H_g^{\sharp}$ is Hermitian conjugated matrix, one finds that $\sigma(H_g) \subset \mathbf{R}$ and therefore the above-referred ordering is correctly defined. Now we can introduce the probability density $P_k(x_{(k)})$ for kth eigenvalue $x_{(k)}$ and the level density

$$\omega_N(x) = \frac{1}{N}\sum_{k=1}^{N} P_k(x). \tag{2}$$

Unfortunately, the famous Semi-Circle Law (derived in [15] for the level density of classical random matrix ensembles) is not applicable for DUE matrices. Instead of a circular shape the level density of damped matrices conforms to a probability density taken from two-parametric family

$$q_{\vartheta,\varepsilon}(x) = \frac{\zeta(\vartheta)}{\varepsilon}\begin{cases} \exp[-\frac{\vartheta^2\varepsilon^2}{\varepsilon^2 - x^2}] & \ldots\ |x| < \varepsilon, \\ 0 & \ldots\ |x| \geqslant \varepsilon, \end{cases} \tag{3}$$

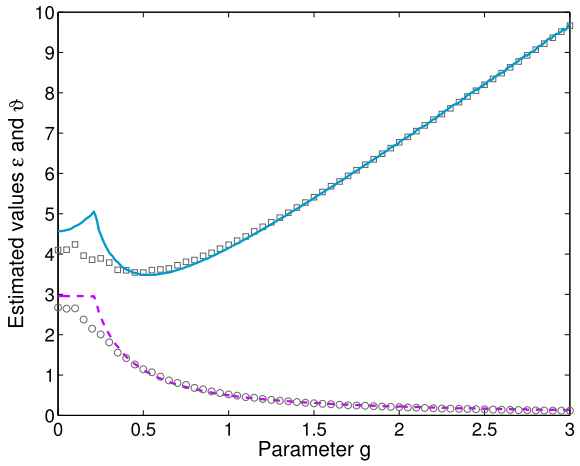

Fig. 2. Estimates of the critical eigenvalue ε and curvature ϑ in (3). A curve and dashed line correspond to $\hat{\varepsilon}$ and $\hat{\vartheta}$ obtained by CSE method. Squares and circles display the estimates $\hat{\varepsilon}$ and $\hat{\vartheta}$ calculated by means of KE.

where $\zeta^{-1}(\vartheta) = \int_{-1}^{1} \exp[-\frac{\vartheta^2}{1-x^2}]\,\mathrm{d}x$ ensures the proper normalization. Indeed, numerical tests show (as is illustrated in Fig. 1) that the level density for large N is very accurately approximated by the function $q_{\vartheta,\varepsilon}(x)$ for estimated values $\hat{\varepsilon} = \hat{\varepsilon}(g,N)$, $\hat{\vartheta} = \hat{\vartheta}(g,N)$ of the so-called critical eigenvalue ε and curvature ϑ. Both the presented estimation-procedures are based on principles of MDE (minimum distance estimation). The first of them (CSE – chi-square estimator) minimizes the statistical distance

$$\chi(\vartheta,\varepsilon) = \sqrt{\int_{-\infty}^{+\infty} |h(x) - q_{\vartheta,\varepsilon}(x)|^2\,\mathrm{d}x}, \tag{4}$$

where $h(x)$ is an empirical histogram. For potential reconstructive purposes we are specifying that the set S consisting of 1 280 000 eigenvalues (taken from 5000 realizations of 256×256 matrices) generates the domain $\mathrm{Dom}(h) = [\min(S), \max(S)]$ of the empirical histogram. In our numerical tests, this domain $\mathrm{Dom}(h)$ has been divided into 200 equidistant sub-intervals.

The second statistical test is based on the standard Kolmogorov estimator (KE) minimizing the supremum of the absolute difference between the estimated distribution function and the empirical distribution function

$$H_N(x) = \frac{1}{N}\sum_{k=1}^{N} \Theta(x - x_{(k)}). \tag{5}$$

Here

$$\Theta(x) = \begin{cases} 1; & x > 0 \\ 0; & x \leqslant 0 \end{cases} \tag{6}$$

stands for the Heaviside step-function. The optimal values of estimated parameters are compared graphically in Fig. 2. Marked discrepancies near the origin can be attributed to the following facts. If $g = 0$ then one can trivially express the level density as $\mathrm{e}^{-\frac{x^2}{2}}/\sqrt{2\pi}$ and our estimation procedure tries, in fact, to approximate the normal distribution by a function $q_{\vartheta,\varepsilon}(x)$. Moreover,

$$\forall \vartheta > 0: \quad \lim_{\varepsilon \to 0_+} q_{\vartheta,\varepsilon}(x) = \lim_{\varkappa \to 0_+} \frac{1}{\sqrt{2\pi}\,\varkappa}\mathrm{e}^{-\frac{x^2}{2\varkappa^2}} = \delta(x),$$

where $\delta(x) \in \mathscr{D}'$ is the Dirac function (see Appendix A.1). Since both these distributions tend (for small values of parameters) to the same generalized limit it is difficult to distinguish them numerically. It means that estimations used (applied for small values