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An algebraic approach to the Hubbard model

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ABSTRACT

We study the algebraic structure of an integrable Hubbard–Shastry type lattice model associated with the centrally extended $\mathfrak{su}(2|2)$ superalgebra. This superalgebra underlies Beisert's AdS/CFT worldsheet R-matrix and Shastry's R-matrix. The considered model specializes to the one-dimensional Hubbard model in a certain limit. We demonstrate that Yangian symmetries of the R-matrix specialize to the Yangian symmetry of the Hubbard model found by Korepin and Uglov. Moreover, we show that the Hubbard model Hamiltonian has an algebraic interpretation as the so-called secret symmetry. We also discuss Yangian symmetries of the A and B models introduced by Frolov and Quinn.

1. Introduction

Exactly solvable models of strongly correlated electrons are of great importance in theoretical condensed matter physics. For instance, they play a prominent role in understanding high-T_c superconductivity. The key example of such a model is the one-dimensional Hubbard model introduced in [1]. It describes the dynamics of interacting electrons in a one-dimensional lattice that models the conduction band of a solid. Each site in the lattice can have four different states. It can be vacant, occupied by a spin up or down electron or occupied by an electron pair.

The Hubbard Hamiltonian $\mathcal{H} = \imath \sum_i \mathcal{K}_{ii+1} + \hbar \sum_i \mathcal{V}_i$ for the infinite lattice is written in terms of the usual creation and annihilation operators $\mathsf{c}^\dagger_{i\alpha}$, $\mathsf{c}_{i\alpha}$, with $\alpha = \uparrow, \downarrow$, of electrons with spin up and down. Here

$$\mathcal{K}_{ii+1} := \sum_{\alpha=\uparrow,\downarrow} (\mathsf{c}^\dagger_{i\alpha}\mathsf{c}_{i+1,\alpha} + \mathsf{c}^\dagger_{i+1,\alpha}\mathsf{c}_{i\alpha}), \ \mathcal{V}_i := (\mathsf{n}_{i\uparrow} - \tfrac{1}{2})(\mathsf{n}_{i\downarrow} - \tfrac{1}{2}),$$

where $\mathsf{n}_{i\alpha} := \mathsf{c}^\dagger_{i\alpha}\mathsf{c}_{i\alpha}$ is the number operator, $i \in \mathbb{N}$ enumerates lattice sites, $\imath := \sqrt{-1}$ and $\hbar$ is the coupling constant.

The integrability of the Hubbard model is known since the works of Lieb and Wu [2]. However, the R-matrix was only found much later by Shastry [3]. Given an integrable model, the R-matrix can usually be found from the underlying symmetries of the model. Such symmetries of the Hubbard model were unknown until very recently.

The Hubbard model was long known to have an exact $\mathfrak{so}(4) = (\mathfrak{su}(2) \oplus \mathfrak{su}(2))/\mathbb{Z}_2$ symmetry [4,5] that can be extended to a certain Yangian [6]. Nevertheless, this was not sufficient to determine Shastry's R-matrix and it was suspected that there are more symmetries underlying the Hubbard model.

The answer to this question of the full symmetry algebra of the Hubbard model came from an unexpected area – the superstring theory. The R-matrix of the Hubbard model emerges in the prime example of the AdS/CFT correspondence as the worldsheet R-matrix [7–10]. It is the intertwining matrix for fundamental representations of the centrally extended $\mathfrak{su}(2|2)$ superalgebra. We will denote this superalgebra by $\mathfrak{g}$. Interestingly, this superalgebra has a non-standard Yangian extension, which is also a symmetry of the R-matrix [11]. An exceptional feature of this Yangian is that it has an additional generator, which has no counterpart in $\mathfrak{g}$, called the secret symmetry [12]. The complete Yangian symmetry is an infinite-dimensional superalgebra of a novel type [13].

In this letter we consider an integrable one-dimensional lattice model by identifying each site of the lattice with the fundamental module of $\mathfrak{g}$. This gives an integrable Hubbard–Shastry type model which specializes to the Hubbard model in a certain limit. Such generalized models were first considered in [14] and [15]. We call this model the general model. (For an overview see Section 12.4 in [16].) Likewise, there are three other parity invariant lattice models that follow from the general model – the A and B models of Frolov and Quinn [17] and the Essler–Korepin–Schoutens (EKS) model [18].

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We construct an oscillator realization of $\mathfrak{g}$, its Yangian extension and the R-matrix, and discuss the integrable structure of the general model and its specializations. In particular we shed more light on the symmetries of the Hubbard model and obtain the spin Yangian symmetry of the Hubbard model [6] as a certain specialization of the Yangian of $\mathfrak{g}$. Moreover, we show that the Hubbard Hamiltonian is a specialization of the secret symmetry. We also obtain Yangian symmetries of the A and B models.

This letter is organized as follows. We first introduce the necessary notation, the setup of the lattice, and the algebra $\mathfrak{g}$. We then construct an oscillator realization of $\mathfrak{g}$ and its Yangian extension, which allows us to compute the R-, Lax- and transfer matrices of the general model. In the remaining sections we discuss links with the A, B and EKS models, the Hubbard model limit, and prove the aforementioned results.

2. Oscillators and vector spaces

Oscillator algebra We consider a one-dimensional lattice with $L \in \mathbb{N}$ sites that can be occupied by spin-$\frac{1}{2}$ particles. Each lattice site is identified with a four-dimensional $\mathbb{Z}_2$-graded vector space $V_i \cong \mathbb{C}^{2|2}$ spanned by vectors

$$V_i = \mathrm{span}_{\mathbb{C}}\{\,|0\rangle_i,\ |{\downarrow}\rangle_i,\ |{\uparrow}\rangle_i,\ |{\uparrow\downarrow}\rangle_i\,\},$$

where $1 \le i \le L$ is the index of the site in the lattice. The entire lattice is the L-fold tensor product $V := \bigotimes_{i=1}^{L} V_i$. The vector $|0\rangle_i$ is the vacuum (i.e. a hole at the ith site), $|{\uparrow}\rangle_i$ and $|{\downarrow}\rangle_i$ are spin up and spin down particles, and $|{\uparrow\downarrow}\rangle_i$ is a multi-particle state occupied by a pair of spin up and spin down particles. The grading is 1 for vectors $|{\uparrow}\rangle_i$ and $|{\downarrow}\rangle_i$, and 0 otherwise. To describe such a lattice, we introduce the usual free-fermion oscillator algebra generated by $\mathsf{c}^\dagger_{i\alpha}$ and $\mathsf{c}_{i\alpha}$ with $1 \le i, j \le L$ and $\alpha, \beta = \uparrow, \downarrow$, that satisfy the standard (anti)commutation relations

$$\{\mathsf{c}_{i\alpha}, \mathsf{c}_{j\beta}\} = 0, \quad \{\mathsf{c}^\dagger_{i\alpha}, \mathsf{c}^\dagger_{j\beta}\} = 0, \quad \{\mathsf{c}^\dagger_{i\alpha}, \mathsf{c}_{j\beta}\} = 1\,\delta_{ij}\delta_{\alpha\beta}.$$

We will denote the universal enveloping algebra of a single copy the oscillator algebra by Osc_i. Likewise, $\mathrm{Osc}_{ij(k\ldots)}$ will denote the universal enveloping algebra of $ij(k\ldots)$ copies of the oscillator algebra. For ease of notation, we identify the unit 1 of the oscillator algebra with the unit 1 of the ground field $\mathbb{C}$. Upon identification

$$|\alpha\rangle_i := \mathbf{e}^{\imath\phi_\alpha i}\mathsf{c}^\dagger_{i\alpha}|0\rangle_i, \quad |{\uparrow\downarrow}\rangle_i := \mathbf{e}^{\imath(\phi_\uparrow+\phi_\downarrow)i}\mathsf{c}^\dagger_{i\uparrow}\mathsf{c}^\dagger_{i\downarrow}|0\rangle_i \tag{2.1}$$

and requiring $\mathsf{c}_{i\alpha}|0\rangle_i = 0$, the space V_i becomes a left Osc_i-module; here $\imath := \sqrt{-1}$ and $\phi_\alpha \in \mathbb{C}$ are arbitrary phases.

Matrix representation Let $E_{ab} \in \mathrm{End}(V_i)$, with $1 \le a, b \le 4$, be the usual graded matrix units satisfying

$$[E_{ab}, E_{cd}] = \delta_{bc}E_{ad} - (-1)^{(\bar a+\bar b)(\bar c+\bar d)}\delta_{ad}E_{cb},$$

$$(E_{ab} \otimes E_{cd})(E_{ij} \otimes E_{kl}) = (-1)^{(\bar c+\bar d)(\bar i+\bar j)}E_{ab}E_{ij} \otimes E_{cd}E_{kl},$$

where $\bar 1 = \bar 4 = 0$ and $\bar 2 = \bar 3 = 1$ is the grading. A matrix representation of the oscillator algebra is given by the map $\pi_i : \mathrm{Osc}_i \to \mathrm{End}(V_i)$ so that

$$\begin{aligned}
&\mathsf{c}_{i\downarrow} \mapsto \mathbf{e}^{\imath\phi_\downarrow i}(E_{12} - E_{34}), && \mathsf{c}_{i\uparrow} \mapsto \mathbf{e}^{\imath\phi_\uparrow i}(E_{13} + E_{24}),\\
&\mathsf{c}^\dagger_{i\downarrow} \mapsto \mathbf{e}^{-\imath\phi_\downarrow i}(E_{21} - E_{43}), && \mathsf{c}^\dagger_{i\uparrow} \mapsto \mathbf{e}^{-\imath\phi_\uparrow i}(E_{31} + E_{42}),\\
&\mathsf{n}_{i\downarrow} \mapsto E_{22} + E_{44}, && \mathsf{n}_{i\uparrow} \mapsto E_{33} + E_{44}.
\end{aligned} \tag{2.2}$$

The additional phases ϕ_α in the expressions above are required to make the Hamiltonian and the R-matrix of the model manifestly invariant (i.e. without additional twists or similarity transformations) under the standard spin and charge symmetries. This will be explained in detail in Section 4. For subsequent reference we also set $(-)^\downarrow := 1$ and $(-)^\uparrow := -1$, $(-)^{\uparrow\downarrow} := -1$.

3. Superalgebra

Superalgebra We recall the definition of $\mathfrak{g}$ due to [8]. By $[\cdot,\cdot]$ we will denote the graded commutator $[a, b] = ab - (-1)^{\bar a\bar b}ba$ for any elements a and b in the superalgebra. The bar $^-: \mathfrak{g} \to \mathbb{Z}_2$ denotes the degree of the element under the $\mathbb{Z}_2$-grading. We will call grade 0 elements bosonic. Likewise, grade 1 elements will be called fermionic.

The centrally extended superalgebra $\mathfrak{su}(2|2)$ is generated by elements E_a, F_a, H_a, with $a = 1, 2, 3$, and central elements P and K subject to the following defining relations:

$$\begin{aligned}
&[H_a, E_b] = A_{ab}E_b, \quad [H_i, F_j] = -A_{ab}F_b,\\
&[E_a, F_b] = \delta_{ab}D_aH_a, \quad [E_1, E_3] = 0, \quad [F_1, F_3] = 0,\\
&[E_c, [E_c, E_2]] = 0, \quad [F_c, [F_c, F_2]] = 0,\\
&[[E_1, E_2], [E_3, E_2]] = P, \quad [[F_1, F_2], [F_3, F_2]] = K
\end{aligned} \tag{3.1}$$

for $a, b = 1, 2, 3$ and $c = 1, 3$. Here $D = \mathrm{diag}(1, -1, -1)$ is the normalization matrix and A is the Cartan matrix given in the appendix. Dynkin nodes 1 and 3 are bosonic; Dynkin node 2 is fermionic. Consequently, the $\mathbb{Z}_2$-grading is 1 for E_2, F_2 and is 0 otherwise.

Local oscillator realization Fix a complex number $\hbar$ (this will be the coupling constant of the model). To each site we associate a spectral parameter u_i, so that the lattice is parametrized by $\vec u = (u_1, u_2, \ldots, u_L)$ and $\hbar$.

To obtain a realization of $\mathfrak{g}$ we introduce a function $x(u)$ and parameters $x_i^\pm$ defined by [7]

$$x(u_i) := \frac{1}{2}\left(u_i + \sqrt{u_i^2 - 4}\right), \quad x_i^\pm := x(u_i \pm \tfrac{1}{2}\hbar)^{\pm 1}. \tag{3.2}$$

The parameters $x_i^\pm$ are Zhukovsky variables satisfying

$$x_i^+ + \frac{1}{x_i^+} - x_i^- - \frac{1}{x_i^-} = \hbar, \quad x_i^+ + \frac{1}{x_i^+} + x_i^- + \frac{1}{x_i^-} = 2u_i.$$

Then, for each site, we introduce the local weights

$$a_i = \frac{\gamma_i}{\sqrt{\hbar}}, \quad b_i = \frac{\nu_i^2 - 1}{\sqrt{\hbar}\,\gamma_i}, \quad c_i = \frac{\gamma_i}{\sqrt{\hbar}\,x_i^+}, \quad d_i = \frac{x_i^+(\nu_i^2 - 1)}{\sqrt{\hbar}\,\nu_i^2\gamma_i},$$

where $\nu_i^2 = x_i^+/x_i^-$ and $\gamma_i^2 = \nu_i(x_i^+ - x_i^-)$. Local weights describe the fundamental representation of $\mathfrak{g}$ at each site.[1]

A local oscillator realization of $\mathfrak{g}$ is given by the map $\eta_i : \mathfrak{g} \to \mathrm{Osc}_i$ such that (for $a = 1, 2, 3$)

$$\begin{aligned}
&E_a \mapsto \mathcal{E}_{i,a}, \quad F_a \mapsto \mathcal{F}_{i,a}, \quad H_a \mapsto \mathcal{H}_{i,a},\\
&P \mapsto \mathcal{P}_i, \quad K \mapsto \mathcal{K}_i, \quad C \mapsto \mathcal{C}_i,
\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
&\mathcal{E}_{i,1} = \mathbf{e}^{\imath(\phi_\uparrow+\phi_\downarrow)i}\mathsf{c}^\dagger_{i\uparrow}\mathsf{c}^\dagger_{i\downarrow}, \quad \mathcal{F}_{i,1} = \mathbf{e}^{-\imath(\phi_\uparrow+\phi_\downarrow)i}\mathsf{c}_{i\downarrow}\mathsf{c}_{i\uparrow},\\
&\mathcal{E}_{i,3} = \mathbf{e}^{\imath(\phi_\uparrow-\phi_\downarrow)i}\mathsf{c}^\dagger_{i\uparrow}\mathsf{c}_{i\downarrow}, \quad \mathcal{F}_{i,3} = \mathbf{e}^{\imath(\phi_\downarrow-\phi_\uparrow)i}\mathsf{c}^\dagger_{i\downarrow}\mathsf{c}_{i\uparrow},\\
&\mathcal{E}_{i,2} = \mathbf{e}^{-\imath\phi_\uparrow i}((a_i - b_i)\,\mathsf{c}_{i\uparrow}\mathsf{n}_{i\downarrow} + b_i\,\mathsf{c}_{i\uparrow}), \quad \mathcal{P}_i = a_ib_i,\\
&\mathcal{F}_{i,2} = \mathbf{e}^{\imath\phi_\uparrow i}((d_i - c_i)\,\mathsf{c}^\dagger_{i\uparrow}\mathsf{n}_{i\downarrow} + c_i\,\mathsf{c}^\dagger_{i\uparrow}), \quad \mathcal{K}_i = c_id_i,\\
&\mathcal{H}_{i,1} = \mathsf{n}_{i\uparrow} + \mathsf{n}_{i\downarrow} - 1, \quad \mathcal{H}_{i,3} = \mathsf{n}_{i\downarrow} - \mathsf{n}_{i\uparrow},\\
&\mathcal{H}_{i,2} = -\mathcal{C}_i - \tfrac{1}{2}\mathcal{H}_{i,1} - \tfrac{1}{2}\mathcal{H}_{i,3}, \quad \mathcal{C}_i = \tfrac{1}{2}(a_id_i + b_ic_i)
\end{aligned} \tag{3.4}$$

[1] The fundamental representation of $\mathfrak{g}$ is a four-dimensional atypical cyclic (non-highest weight) representation. See sections 2.3 and 2.4 in [8] for more details.