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# Paths of zeros of analytic functions describing finite quantum systems

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Quantum systems with positions and momenta in $\mathbb{Z}(d)$ are described by the $d$ zeros of analytic functions on a torus. The $d$ paths of these zeros on the torus describe the time evolution of the system. A semi-analytic method for the calculation of these paths of the zeros is discussed. Detailed analysis of the paths for periodic systems is presented. A periodic system which has the displacement operator to a real power $t$, as time evolution operator, is studied. Several numerical examples, which elucidate these ideas, are presented.



## 1. Introduction

There is an extensive literature on analytic representations in quantum mechanics, after the pioneering work by Bargmann [1,2]. The Bargmann analytic function in the complex plane studies problems related to the harmonic oscillator. The zeros of the Bargmann function, which are also the zeros of the Husimi (or $Q$) function, provide a valuable insight to various quantum systems [3–10], chaos [10], etc. Other potential applications include the study of two-dimensional electron gas in a magnetic field, quantum Hall effect, [11–13], etc.

Analytic representations in the unit disc for problems with $SU(1,1)$ symmetry, and analytic representations in the extended complex plane for systems with $SU(2)$ symmetry, have also been studied in the literature (reviews have been presented in [14–16]).

Quantum systems with variables in $\mathbb{Z}(d)$ (the integers modulo $d$) have been studied extensively in the literature (e.g., [17–20]). Refs. [21–25] have represented their quantum states with analytic functions on a torus, using Theta functions. It has been shown that these functions have exactly $d$ zeros, which determine uniquely the state of the system. As the system evolves in time, the zeros follow $d$ paths, on the torus. Ref. [4] has also used a similar representation in studies of chaos. Theta functions have been used extensively in various problems in physics [26,27].

In this paper we study different aspects of the zeros of analytic functions for finite quantum systems with variables in $\mathbb{Z}(d)$, as follows:

- We propose in Eqs. (18), (19) a semi-analytic method for the calculation of the paths of the zeros, which is primarily analytical (section 2). Previous work is based on entirely numerical methods. In principle the full quantum formalism can be expressed in terms of the $d$ zeros. But it is difficult to express physical laws in terms of the zeros, without an analytical formalism that relates physical quantities to the zeros. The semi-analytical formalism in this paper is a step in this direction.
- We study in detail the $d$ paths of the zeros of periodic systems. Each path is characterized by the multiplicity M, and by a pair of winding numbers $(w_1, w_2)$. An interesting periodic system is one, which has as time evolution operator the displacement operator to a real power $t$. Displacement operators $Z^\alpha \mathcal{X}^\beta$ are defined in finite quantum systems for $\alpha, \beta \in \mathbb{Z}(d)$, and it is interesting to study these operators to a real power $t$. It is shown that the paths of the zeros are identical, but shifted with respect to each other (section 3).

## 2. Analytic representation of finite quantum systems

We consider a finite quantum system with variables in $\mathbb{Z}(d)$. This system is described with the $d$-dimensional Hilbert space $\mathcal{H}(d)$. Let $|X;m\rangle$ and $|P;m\rangle$ (where $m \in \mathbb{Z}(d)$) be the position and momentum bases which are related through a Fourier transform, as follows:

$$|P;n\rangle = \mathcal{F}|X;n\rangle; \qquad \mathcal{F} = d^{-1/2}\sum_{m,n}\omega(mn)|X;m\rangle\langle X;n|;$$

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$$\omega(m) = \exp\left[i\frac{2\pi m}{d}\right] \tag{1}$$

Let $|g\rangle$ be an arbitrary state

$$|g\rangle = \sum_m g_m|X;m\rangle; \qquad \sum_m |g_m|^2 = 1 \tag{2}$$

We use the notation

$$|g^*\rangle = \sum_m g_m^*|X;m\rangle; \quad \langle g| = \sum_m g_m^*\langle X;m|$$
$$\langle g^*| = \sum_m g_m\langle X;m| \tag{3}$$

We represent the state $|g\rangle$ with the analytic function [3,4,21]

$$G(z) = \pi^{-1/4}\sum_{m=0}^{d-1} g_m\,\Theta_3\left[\frac{\pi m}{d} - z\sqrt{\frac{\pi}{2d}};\frac{i}{d}\right] \tag{4}$$

where $\Theta_3$ is the Theta function [28]

$$\Theta_3(u,\tau) = \sum_{n=-\infty}^{\infty}\exp(i\pi\tau n^2 + i2nu)$$
$$\Theta_3'(u,\tau) = \frac{\mathrm{d}\Theta_3}{\mathrm{d}u} = i\sum_{n=-\infty}^{\infty} 2n\exp(i\pi\tau n^2 + i2nu). \tag{5}$$

We can prove that

$$G(z+\sqrt{2\pi d}) = G(z)$$
$$G(z+i\sqrt{2\pi d}) = G(z)\exp\left(\pi d - iz\sqrt{2\pi d}\right), \tag{6}$$

and therefore it is sufficient to have this function in a cell

$$S = [M\sqrt{2\pi d},\,(M+1)\sqrt{2\pi d}) \times [N\sqrt{2\pi d},\,(N+1)\sqrt{2\pi d}) \tag{7}$$

where $(M,N)$ are integers labelling the cell. Other models with more general quasi-periodic boundary conditions can also be studied. The scalar product is given by

$$\langle f^*|g\rangle = \frac{1}{d^{3/2}\sqrt{2\pi}}\int_S \mathrm{d}\mu(z)F(z^*)G(z) = \sum f_m g_m;$$
$$\mathrm{d}\mu(z) = \mathrm{d}^2z\exp\left(-z_I^2\right) \tag{8}$$

These relations are proved using the orthogonality relation [22]

$$2^{-1/2}\pi^{-1}d^{-3/2}\int_S \mathrm{d}\mu(z)\Theta_3\left[\frac{\pi n}{d} - z\sqrt{\frac{\pi}{2d}};\frac{i}{d}\right]$$
$$\times\,\Theta_3\left[\frac{\pi m}{d} - z^*\sqrt{\frac{\pi}{2d}};\frac{i}{d}\right] = \delta(m,n) \tag{9}$$

The coefficients $g_m$ in Eq. (2) are given by

$$g_m = 2^{-1/2}\pi^{-3/4}d^{-3/2}\int_S \mathrm{d}\mu(z)\Theta_3\left[\frac{\pi m}{d} - z\sqrt{\frac{\pi}{2d}};\frac{i}{d}\right]G(z^*). \tag{10}$$

It has been proved in [4,21] that the analytic function $G(z)$ has exactly $d$ zeros $\zeta_n$ in each cell $S$, and that

$$\sum_{n=1}^{d}\zeta_n = \sqrt{2\pi d}(M+iN) + d^{3/2}\sqrt{\frac{\pi}{2}}(1+i). \tag{11}$$

In finite systems the $d-1$ zeros define uniquely the state (the last zero is determined from Eq. (11)). In infinite systems the zeros do not define uniquely the state.

If the $d-1$ zeros $\zeta_n$ are given, the last one can be found from Eq. (11), and the function $G(z)$ is given by

$$G(z) = \mathcal{N}(\{\zeta_n\})$$
$$\times\exp\left[-i\sqrt{\frac{2\pi}{d}}Nz\right]\prod_{n=1}^{d}\Theta_3$$
$$\times\left[\sqrt{\frac{\pi}{2d}}(z-\zeta_n) + \frac{\pi(1+i)}{2};\ i\right] \tag{12}$$

Here $N$ is the integer that labels the cell (as in Eq. (7)), and $\mathcal{N}(\{\zeta_n\})$ is a normalization constant that does not depend on $z$ (see section 7 in Ref. [21]). Below we choose the cell with $M=N=0$.

### *2.1. Time evolution and paths of zeros*

Let $H$ be the Hamiltonian of the system (a $d\times d$ Hermitian matrix $H_{mn}$). As the system evolves in time $t$, each zero $\zeta_n$ follows a path $\zeta_n(t)$.

We consider infinitesimal changes to the coefficients from $g_m$ to $g_m+\Delta g_m$, where

$$\Delta g_m = i\Delta t\sum_n H_{mn}g_n \tag{13}$$

Then the zeros will change from $\zeta_n$ to $\zeta_n+\Delta\zeta_n$. From Eqs. (4), (12) we get

$$\pi^{-1/4}\sum_{m=0}^{d-1}(g_m+\Delta g_m)\,\Theta_3\left[\frac{\pi m}{d} - z\sqrt{\frac{\pi}{2d}};\frac{i}{d}\right]$$
$$= \mathcal{N}(\{\zeta_k\})\prod_{n=1}^{d}\Theta_3\left[\sqrt{\frac{\pi}{2d}}(z-\zeta_n-\Delta\zeta_n) + \frac{\pi(1+i)}{2};\ i\right] \tag{14}$$

With a Taylor expansion of the right hand side, we get

$$\pi^{-1/4}\sum_{m=0}^{d-1}\Delta g_m\,\Theta_3\left[\frac{\pi m}{d} - z\sqrt{\frac{\pi}{2d}};\frac{i}{d}\right] = -\mathcal{N}(\{\zeta_k\})\sqrt{\frac{\pi}{2d}}$$
$$\times\sum_{j=1}^{d}A_j(z)\Theta_3'\left[\sqrt{\frac{\pi}{2d}}(z-\zeta_j) + \frac{\pi(1+i)}{2};\ i\right]\Delta\zeta_j$$
$$A_j(z) = \prod_{m\neq j}\Theta_3\left[\sqrt{\frac{\pi}{2d}}(z-\zeta_m) + \frac{\pi(1+i)}{2};\ i\right] \tag{15}$$

We insert $z=\zeta_n$ on both sides of this equation. For $j\neq n$ we get $A_j(\zeta_n)=0$. Therefore

$$\pi^{-1/4}\sum_{m=0}^{d-1}\Delta g_m\,\Theta_3\left[\frac{\pi m}{d} - \zeta_n\sqrt{\frac{\pi}{2d}};\frac{i}{d}\right]$$
$$= -\mathcal{N}(\{\zeta_k\})\sqrt{\frac{\pi}{2d}}A_n(\zeta_n)\Theta_3'\left[\frac{\pi(1+i)}{2};\ i\right]\Delta\zeta_n$$
$$A_n(\zeta_n) = \prod_{m\neq n}\Theta_3\left[\sqrt{\frac{\pi}{2d}}(\zeta_n-\zeta_m) + \frac{\pi(1+i)}{2};\ i\right] \tag{16}$$

Using Eq. (5), we found numerically that

$$\Theta_3'\left[\frac{\pi(1+i)}{2};\ i\right] = 1.9888i. \tag{17}$$