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Anisotropic metamaterial as an analogue of a black hole

Isabel Fernández-Núñez, Oleg Bulashenko *

Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain

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A B S T R A C T

Propagation of light in a metamaterial medium which mimics curved spacetime and acts like a black hole is studied. We show that for a particular type of spacetimes and wave polarization, the time dilation appears as dielectric permittivity, while the spatial curvature manifests as magnetic permeability. The optical analogue to the relativistic Hamiltonian which determines the ray paths (null geodesics) in the anisotropic metamaterial is obtained. By applying the formalism to the Schwarzschild metric, we compare the ray paths with full-wave simulations in the equivalent optical medium.

1. Introduction

One of the hot topics of modern technology is to build artificial materials whose permittivity and permeability can be properly engineered by incorporating structural elements of subwavelength sizes. As a result, one can create materials (called *metamaterials*) with the desired electromagnetic response which offers new opportunities for realizing such exotic phenomena as negative refraction, cloaking, super-lenses for subwavelength imaging, microantennas, etc. [1,2].

Recently, it has also been recognized that metamaterials can be used to mimic general-relativity phenomena [3,4]. The propagation of electromagnetic waves in curved spacetime is formally equivalent to the propagation in flat spacetime in a certain inhomogeneous anisotropic or bianisotropic medium [5–12]. Based on this equivalence, different general relativity phenomena have been discussed from the point of view of possible realization in metamaterials: optical analogues of black holes [13–18], Schwarzschild spacetime [12,19], de Sitter spacetime [20–22], cosmic strings [23,24], wormholes [25], Hawking radiation [26], the "Big Bang" and cosmological inflation [27,28], colliding gravitational waves [29], among others.

The deflection of light waves in gradient-index optical materials mimicking optical black holes was studied theoretically [13,14,18] and experimentally [15–18]. These materials, called by authors "omnidirectional electromagnetic absorbers", are characterized by an isotropic effective refractive index. A real cosmological black hole (BH) can often be described by an anisotropic spacetime, as, for example, the case of the Schwarzschild BH. For that case, one should determine the permittivity and permeability tensors instead of the refractive index in order to introduce the equivalent optical medium [12,19]. Chen et al. [19] simulated the wave propagation outside the Schwarzschild BH and observed in their numerical results the phenomenon of "photon sphere", which is an important feature of the BH system. It would be interesting to go further and study the propagation of light waves in optically anisotropic media which mimic cosmological BHs and compare the results with ray paths obtained from the Hamiltonian method.

The aim of this letter is twofold. First, we determine the constitutive relations of an inhomogeneous anisotropic medium which is formally equivalent to the static spacetime metric obeying rotational symmetries and can be applied, in principle, to the medium either in isotropic or anisotropic form. Second, by making use of the eikonal approximation to the wave equation, we obtain the expression for the optical Hamiltonian which we found to be identical to the one obtained from general relativity for null geodesics, but different from the optical Hamiltonian used in Refs. [13,15,30,31]. Then we apply the formalism to the Schwarzschild spacetime that is a solution to the Einstein field equations in vacuum [32]. We compare the wave propagation with the ray dynamics outside the BH in the effective medium and obtain a very good correspondence. As an interesting feature we find that light does not propagate in the direction of the wave normal, there is an angle between the wave velocity and the ray velocity. The obtained

* Corresponding author at: Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain. Tel.: +34 934037103.
E-mail address: oleg.bulashenko@ub.edu (O. Bulashenko).

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results are discussed from the point of view of metamaterial implementation.

2. General relativity in a metamaterial medium

2.1. Medium parameters

Long time ago, Tamm pointed out the parallels between anisotropic crystals and curved spacetimes [5]. Later studies showed [6–11] that the propagation of light in empty curved space distorted by a gravitational field is formally equivalent to light propagation in flat space filled with an inhomogeneous anisotropic medium.

Indeed, consider a spacetime background with a general metric[1]

$$\mathrm{d}s^2 = g_{00}\,\mathrm{d}t^2 + 2g_{0i}\,\mathrm{d}t\,\mathrm{d}x^i + g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j, \tag{1}$$

where $i, j = 1, 2, 3$ run over arbitrary spatial coordinates. Then, it can be shown [11] that the covariant Maxwell's equations written in curved coordinates can be transformed into their standard form for flat space but in the presence of an effective medium. The constitutive relations of the equivalent medium have been found in the form [7]:

$$D^i = \varepsilon^{ij} E_j - (\mathbf{\Gamma} \times \mathbf{H})^i, \quad B^i = \mu^{ij} H_j + (\mathbf{\Gamma} \times \mathbf{E})^i, \tag{2}$$

which connect the fields **D**, **B**, **E** and **H** via nontrivial permittivity and permeability tensors

$$\varepsilon^{ij} = \mu^{ij} = -\frac{\sqrt{-g}}{g_{00}} g^{ij} \tag{3}$$

and a vector $\mathbf{\Gamma}$ given by

$$\Gamma_i = -\frac{g_{0i}}{g_{00}}. \tag{4}$$

Here, g^{ij} is the inverse of g_{ij} and g is the determinant of the full spacetime metric $g_{\mu\nu}$, with $\mu, \nu = 0, 1, 2, 3$. Note that the information about the gravitational field is essentially embedded in the material properties of the effective medium: the tensors ε^{ij}, μ^{ij} which are symmetric and should be equal, and the vector $\mathbf{\Gamma}$ which couples the electric and magnetic fields. The invention of metamaterials during the last decade [1,2] opened up the possibility to design electromagnetic media corresponding to different spacetimes [3,4,19–29].

In this letter, we consider a static spacetime metric associated with a spherically symmetric cosmological BH. Due to time-reversal symmetry, $g_{0i} = 0$ and the coupling between the electric and magnetic fields vanishes, $\mathbf{\Gamma} = 0$. The metric (1) in (t, r, θ, φ) coordinates can then be written in a generic form as [32]

$$\mathrm{d}s^2 = g_{00}(r)\,\mathrm{d}t^2 + g_{rr}(r)\left\{\mathrm{d}r^2 + f(r)\,[r^2\,\mathrm{d}\theta^2 + r^2 \sin^2\theta\,\mathrm{d}\varphi^2]\right\}, \tag{5}$$

where f is the "anisotropic factor". Note that the metric (5) obeys the rotational symmetries in the three-dimensional (r, θ, φ) space.

Then, we have to project the metric (5) into a flat background to obtain the medium parameters in the Cartesian coordinate system. To do that, we apply a coordinate transformation and, from Eq. (3), we get the permittivity and permeability tensors in the form:

$$\varepsilon^{ij} = \mu^{ij} = \sqrt{-\frac{g_{rr}}{g_{00}}}\left[\delta^{ij} - (1-f)\frac{x^i x^j}{r^2}\right], \tag{6}$$

where δ^{ij} is the Kronecker delta, $r = \sqrt{x^2 + y^2 + z^2}$, and we denoted the Cartesian coordinates $(x^1, x^2, x^3) \equiv (x, y, z)$. It is seen that whenever $f \neq 1$, the permittivity and permeability tensors contain the off-diagonal elements and the equivalent medium is essentially anisotropic. Only in the case of $f = 1$, the off-diagonal elements vanish and the medium becomes completely isotropic with all the diagonal elements equal to the refractive index: $n(r) = \sqrt{-g_{rr}/g_{00}}$. Note that in general relativity the spacetime with $f = 1$ in Eq. (5) is said to be conformal to flat space. Every static spherically symmetric spacetime with $f \neq 1$ can be converted to conformally flat form by an appropriate transformation of the radial coordinate: $r \to \rho$. The new radial coordinate is obtained by

$$\rho = r \exp\left\{\int_r^{\infty}\left[1 - \frac{1}{\sqrt{f(r')}}\right]\frac{\mathrm{d}r'}{r'}\right\}, \tag{7}$$

where the isotropic boundary condition at infinity, $f(\infty) = 1$, is taken into account. The line element in the $(t, \rho, \theta, \varphi)$ *isotropic* coordinates takes the conformally flat form:

$$\mathrm{d}s^2 = g_{00}[r(\rho)]\,\mathrm{d}t^2 + \Lambda(\rho)\,(\mathrm{d}\rho^2 + \rho^2 \sin^2\theta\,\mathrm{d}\varphi^2 + \rho^2 \mathrm{d}\theta^2), \tag{8}$$

where the time dilation term g_{00} and the conformal factor $\Lambda = g_{rr} f r^2/\rho^2$ are calculated by means of the function $r(\rho)$ which should be obtained by inverting (7). Thus, the permittivity and permeability tensors are simply reduced to the isotropic refractive index:

$$\varepsilon^{ij} = \mu^{ij} = \delta^{ij}\sqrt{-\frac{\Lambda}{g_{00}}} \equiv \delta^{ij}\, n(\rho). \tag{9}$$

The equivalent medium determined by Eq. (9) is still inhomogeneous since the refractive index varies with the radial coordinate, but the light velocity in the medium becomes isotropic, a property that is much simpler to implement in metamaterial design.

In what follows, we will compare the results for light propagation in two different equivalent media – isotropic and anisotropic – both corresponding to the same spacetime metric in order to see the physical differences.

2.2. Electromagnetic fields. TE and TM polarizations

The results from the previous section indicate that an electromagnetic field can be thought of as propagating in flat background but in the presence of a medium whose properties are constructed from a curved spacetime. The fields, for the static case we consider, are related by:

$$D^i = \varepsilon^{ij} E_j, \quad B^i = \mu^{ij} H_j. \tag{10}$$

Due to the anisotropy of the medium for the metric in nonconformally flat form, the electric displacement field **D** is not in the direction of **E**, and the magnetic induction field **B** is not in the direction of **H**. To simplify the treatment of the problem, we consider the propagation of light in the equatorial plane, $z = 0$. In such a case, one of the anisotropies – electric or magnetic – can be eliminated.

Indeed, consider the TE polarization for an electromagnetic wave for which **E** is perpendicular to the x–y plane. Equation (6) for $z = 0$ leads to $\varepsilon^{xz} = \varepsilon^{yz} = 0$, hence the directions of **D** and **E** coincide. This means that ε^{zz} is the only relevant matrix element which connects the nonzero electric components of the field: $D^z = \varepsilon^{zz} E_z$, and the electric anisotropy of the medium is irrelevant. As for the magnetic components, we obtain $\mu^{xz} = \mu^{yz} = 0$,

[1] From now on we follow the standard notations for covariant (subindices) and contravariant (superindices) quantities.