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Lie symmetries and their inverse problems of nonholonomic Hamilton systems with fractional derivatives ☆

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ABSTRACT

This letter focuses on studying Lie symmetries and their inverse problems of the fractional nonholonomic Hamilton systems. Based on the invariance of the fractional motion equations, constraint equations and virtual displacement restrictive conditions of the systems under the infinitesimal transformation with respect to the time and generalized coordinates, the Lie symmetries and conserved quantities of the fractional nonholonomic Hamilton system are discussed and the corresponding definitions, determining equations, limiting equations, additional restricting equations and Lie theorems are given. The letter also systematically studies inverse theorems of Lie symmetries of the fractional nonholonomic Hamilton systems. Finally, an example is discussed to illustrate theses results.

1. Introduction

It is well known that the fractional derivative has been applied in physics and engineering [1,2], and becomes one of the most powerful and widely useful tools in describing and explaining some physical complex systems [3–5]. Recently, Riewe investigated the fractional variational problems and obtained the Euler–Lagrange equations for both conservative and non-conservative systems [6,7]. Agrawal presented the formulation of fractional Euler–Lagrange equations by using the left and right fractional derivatives in terms of Riemann–Liouville fractional derivatives [8]. The symmetries and conservation laws [9–21] have also been extended to fractional derivative systems, such as Torres proved a fractional Noether's theorem of Euler–Lagrange equations [22], Malinowska obtained the fractional Noether-type theorem for conservative and nonconservative generalized physical systems [23], Zhou et al. studied the Noether symmetry theories of the fractional Hamiltonian systems [24]. However, previous work so far has been limited to Noether symmetry theorems [25–28]. Sun gave the fractional first-order and second-order extensions form of Lie group transformation, and the corresponding Lie symmetries of fractional nonholonomic systems were discussed [29]. Therefore, the study of fractional Lie symmetries of systems has attracted much attention.

In this letter, we provide the Lie symmetries and their inverse problems of fractional nonholonomic Hamilton systems. The outline of this paper is as follows. In Section 2, we present a brief summary of the definitions and properties in terms of Riemann–Liouville fractional derivatives. Section 3 discusses the equations of motion of fractional nonholonomic Hamilton systems. Section 4 provides a full Lie symmetry of the systems by introducing infinitesimal transformations with respect to time, generalized coordinates and generalized momenta. In Section 5, the inverse problems of Lie symmetries of fractional nonholonomic Hamilton systems are presented. Finally, Section 6 discusses an example to illustrate the above results.

2. Riemann–Liouville fractional derivatives

We present here a brief summary of definitions and properties in terms of Riemann–Liouville fractional derivatives [30].

Definition 1. Let f be a continuous and integrable function in the interval $[a,b]$. For all $t \in [a,b]$, the left Riemann–Liouville fractional derivatives ${}_aD_t^\alpha f(t)$ and the right Riemann–Liouville fractional derivatives ${}_tD_b^\alpha f(t)$ are defined as follows

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\mathrm{d}\tau, \quad (1)$$

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$$ {}_tD_b^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\int_t^b(\tau-t)^{n-\alpha-1}f(\tau)\mathrm{d}\tau, \qquad (2)$$

where α is the order of the derivatives such that $n-1\le\alpha<n$, $n\in\mathbb{N}$, Γ is the Euler gamma function.

Remark 1. If α is an integer, the fractional derivatives (1) and (2) change into the standard derivatives, *i.e.*

$$ {}_aD_t^\alpha f(t)=\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^\alpha f(t),$$
$$ {}_tD_b^\alpha f(t)=\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^\alpha f(t).$$

Theorem 1. *Let f and g be two continuous functions defined on the interval $[a,b]$, and ϕ, φ are constants. Then for all $t\in[a,b]$, the following property holds*

$$ {}_aD_t^\alpha\big[\varphi f(t)+\phi g(t)\big]=\varphi\,{}_aD_t^\alpha f(t)+\phi\,{}_aD_t^\alpha g(t). \qquad (3)$$

Remark 2. In general, the Riemann–Liouville fractional derivative of a constant is not equal to zero. More precisely, one has

$$ {}_aD_t^\alpha c=\frac{c}{\Gamma(1-\alpha)}(t-a)^{-\alpha}. \qquad (4)$$

3. Fractional nonholonomic Hamilton equations

The fractional nonholonomic system is described by n generalized coordinates q_k $(k=1,\cdots,n)$, subjected to the g ideal bilateral fractional nonholonomic constraints of the form

$$ f_\gamma=f_\gamma\big(t,q_k,{}_aD_t^\alpha q_k,{}_tD_b^\beta q_k\big)\quad(\gamma=1,\cdots,g), \qquad (5)$$

which satisfy the Appell–Chetaev's condition

$$ \frac{\partial f_\gamma}{\partial_a D_t^\alpha q_k}\delta q_k=0,$$
$$ \frac{\partial f_\gamma}{\partial_t D_b^\beta q_k}\delta q_k=0\quad(k=1,\cdots,n), \qquad (6)$$

the Einstein summation convention is adopted in this letter.

The fractional nonholonomic system is described by the Lagrange equations [24]:

$$ \frac{\partial L}{\partial q_k}+{}_tD_b^\alpha\frac{\partial L}{\partial_a D_t^\alpha q_k}+{}_aD_t^\beta\frac{\partial L}{\partial_t D_b^\beta q_k}+Q_k''+\lambda_\gamma\frac{\partial f_\gamma}{\partial_a D_t^\alpha q_k}+\lambda_\gamma\frac{\partial f_\gamma}{\partial_t D_b^\beta q_k}=0, \qquad (7)$$

where $L=L(t,q_k,{}_aD_t^\alpha q_k,{}_tD_b^\beta q_k)$ is the fractional Lagrangian, Q_k'' are the nonconservative forces and λ_γ are the Lagrange multipliers.

Eqs. (7) can be written in an equivalent form

$$ \frac{\partial L}{\partial q_k}+{}_tD_b^\alpha\frac{\partial L}{\partial_a D_t^\alpha q_k}+{}_aD_t^\beta\frac{\partial L}{\partial_t D_b^\beta q_k}=-Q_k''-\Lambda_k, \qquad (8)$$

where

$$ \Lambda_k=\Lambda_k\big(t,q_k,{}_aD_t^\alpha q_k,{}_tD_b^\beta q_k\big)=\lambda_\gamma\frac{\partial f_\gamma}{\partial_a D_t^\alpha q_k}+\lambda_\gamma\frac{\partial f_\gamma}{\partial_t D_b^\beta q_k}. \qquad (9)$$

Let us introduce the fractional generalized momenta [31]

$$ p_k^\alpha=\frac{\partial L}{\partial_a D_t^\alpha q_k},\qquad p_k^\beta=\frac{\partial L}{\partial_t D_b^\beta q_k}. \qquad (10)$$

The Hamiltonian depending on the fractional derivatives can be expressed as

$$ H=p_k^\alpha\,{}_aD_t^\alpha q_k+p_k^\beta\,{}_tD_b^\beta q_k-L. \qquad (11)$$

Calculating the total differential of this Hamiltonian, we have

$$ \mathrm{d}H=\mathrm{d}p_k^\alpha\,{}_aD_t^\alpha q_k+p_k^\alpha\mathrm{d}_aD_t^\alpha q_k+\mathrm{d}p_k^\beta\,{}_tD_b^\beta q_k+p_k^\beta\mathrm{d}_tD_b^\beta q_k-\mathrm{d}L. \qquad (12)$$

From Eqs. (8), (10) and (12), one obtains

$$ \mathrm{d}H=\mathrm{d}p_k^\alpha\,{}_aD_t^\alpha q_k+\mathrm{d}p_k^\beta\,{}_tD_b^\beta q_k+\big({}_tD_b^\alpha p_k^\alpha+{}_aD_t^\beta p_k^\beta+Q_k''+\Lambda_k\big)\mathrm{d}q_k-\frac{\partial L}{\partial t}\mathrm{d}t. \qquad (13)$$

This means that the Hamiltonian is a function of the form $H=H(t,q_k,p_k^\alpha,p_k^\beta)$, so we may express Eq. (13) in the form

$$ \mathrm{d}H=\frac{\partial H}{\partial t}\mathrm{d}t+\frac{\partial H}{\partial q_k}\mathrm{d}q_k+\frac{\partial H}{\partial p_k^\alpha}\mathrm{d}p_k^\alpha+\frac{\partial H}{\partial p_k^\beta}\mathrm{d}p_k^\beta. \qquad (14)$$

Comparing Eq. (13) with Eq. (14), we obtain the following fractional Hamilton equations of nonholonomic system

$$ {}_tD_b^\alpha p_k^\alpha+{}_aD_t^\beta p_k^\beta=\frac{\partial H}{\partial q_k}-\tilde{Q}_k''-\tilde{\Lambda}_k,$$
$$ {}_aD_t^\alpha q_k=\frac{\partial H}{\partial p_k^\alpha},\qquad {}_tD_b^\beta q_k=\frac{\partial H}{\partial p_k^\beta},\qquad \frac{\partial H}{\partial t}=-\frac{\partial L}{\partial t}, \qquad (15)$$

where

$$ \tilde{Q}_k''=\tilde{Q}_k''\big(t,q_k,p_k^\alpha,p_k^\beta\big),$$
$$ \tilde{\Lambda}_k=\tilde{\Lambda}_k\big(t,q_k,p_k^\alpha,p_k^\beta\big). \qquad (16)$$

Therefore, the fractional nonholonomic constraints (5) can be written as

$$ \tilde{f}_\gamma=\tilde{f}_\gamma\big(t,q_k,p_k^\alpha,p_k^\beta\big)\quad(\gamma=1,\cdots,g). \qquad (17)$$

4. Lie symmetries of fractional nonholonomic Hamilton systems

Let us introduce the infinitesimal transformations with respect to time, generalized coordinates and generalized momenta

$$ t^*=t+\Delta t,$$
$$ q_k^*(t^*)=q_k(t)+\Delta q_k,$$
$$ (p_k^\alpha)^*(t^*)=p_k^\alpha(t)+\Delta p_k^\alpha,$$
$$ (p_k^\beta)^*(t^*)=p_k^\beta(t)+\Delta p_k^\beta, \qquad (18)$$

and their expansion formulae

$$ t^*=t+\varepsilon\xi\big(t,q_k,p_k^\alpha,p_k^\beta\big)+o(\varepsilon),$$
$$ q_k^*(t^*)=q_k(t)+\varepsilon\eta_k\big(t,q_k,p_k^\alpha,p_k^\beta\big)+o(\varepsilon),$$
$$ (p_k^\alpha)^*(t^*)=p_k^\alpha(t)+\varepsilon\mu_k^\alpha\big(t,q_k,p_k^\alpha,p_k^\beta\big)+o(\varepsilon),$$
$$ (p_k^\beta)^*(t^*)=p_k^\beta(t)+\varepsilon\mu_k^\beta\big(t,q_k,p_k^\alpha,p_k^\beta\big)+o(\varepsilon), \qquad (19)$$

where ε is a small parameter, and ξ, η_k, μ_k^α, μ_k^β are infinitesimal generators.

The relationships between the isochronous variation and the complete variation are given by [32]

$$ \delta q_k=\Delta q_k-\dot{q}_k\Delta t,\qquad \delta_aD_t^\alpha q_k={}_aD_t^\alpha\delta q_k,$$
$$ \delta_tD_b^\beta q_k={}_tD_b^\beta\delta q_k, \qquad (20)$$