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Confinement of an electron in a non-homogeneous magnetic field: Integrable vs superintegrable quantum systems

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ABSTRACT

This paper deals with the problem of an electron in a non-homogeneous magnetic field perpendicular to a plane. From the classical point of view this is an integrable, but not superintegrable, solvable system. In the quantum framework of the Dirac equation this integrable system is solvable too; the energy levels and wavefunctions of bound states, for its reduction to the plane, are computed. The effective one-dimensional matrix Hamiltonian is shown to belong to a shape-invariant hierarchy. Through this example we will shed some light on the specific properties of a quantum integrable system with respect to those characteristic of superintegrable systems.

1. Introduction

This work is devoted to a system that consists of an electron under an external magnetic field perpendicular to the x–y plane. The magnetic field is non-uniform, its intensity behaving as the inverse of the distance to the z-axis. In these conditions the classical system is integrable, but not superintegrable. This system can be restricted to the plane x–y, and in this sense it is quite interesting to find its properties in the light of other well-known superintegrable systems, such as the Landau system of a constant perpendicular magnetic field or the planar Coulomb system. For instance, in our present situation there can exist bounded, although non-periodic, motions or exclusively unbounded motions depending on the sign of the angular momentum. As the trajectories and motion can be obtained in closed implicit form, we can say that the system is solvable.

In the quantum framework of the Dirac equation the system is solvable too, and for such above mentioned sign, the solutions to the eigenvalue problem will be obtained. As the system is solvable, it is investigated whether the reduced radial matrix Hamiltonian belongs to a shape-invariant Hamiltonian hierarchy. In this context, the matrix intertwining operators will be characterized as well as the symmetries of the hierarchy. This implies that the solutions can be obtained algebraically, by means of raising operators. We want to remark that we have found some new properties related to the matrix shape-invariance: (i) It can be realized by means of anti-intertwining operators, due to the fact that the Dirac equation has positive and negative eigenvalues; (ii) There is a wide freedom in the intertwining operators of the shape-invariance in the same hierarchy, in particular we have characterized a four parameter set of such operators; (iii) The symmetries of the shape-invariant hierarchy of matrix Hamiltonians are shown to play an important role.

We will remark the most important properties of the spectrum of bound states of this integrable system with respect to those of superintegrable quantum systems (see for instance the review of Ref. [1] on superintegrability). The most striking difference is that it consists of a dense set of non-isolated points while the known superintegrable systems have a set of isolated points as the discrete spectrum. Contrary to some general belief, the spectrum is highly degenerated, although the system is not superintegrable [1–3]. We will also show how this system is algebraically solvable, but the involved operators have some important differences to those corresponding to superintegrable systems.

This work will develop and extend some methods introduced in a previous paper for a different problem [4]. The present paper is organized as follows. The system is introduced in its classical version along Section 2 showing the features of the classical trajectories. The relativistic quantum system, in the frame of the Dirac equation, is analyzed in Section 3, where the discrete spectrum and eigenfunctions are obtained. In Section 4 the shape-invariant properties of the reduced radial matrix Hamiltonian are investi-

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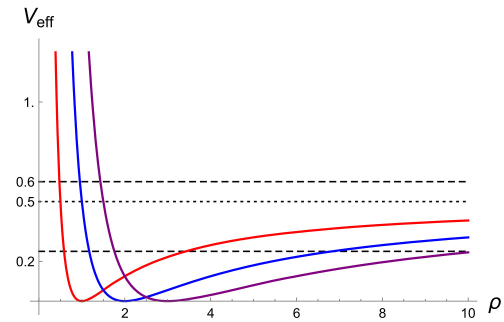

Fig. 1. Effective potential for $\ell = 1, 2, 3$ and $m_0 = c = k = e = 1$, $\mu = 0$. The dashing lines correspond to the energies $E = 0.25$ (bound trajectory) and $E = 0.6$ (unbounded trajectory). The dotted line separates the energies of bound and unbounded motions.

gated. Finally, Section 5 will be devoted to some remarks and conclusions.

2. Classical motion

We will consider an electron under the influence of a magnetic field with a rotational symmetry around the z-axis given by

$$\mathbf{B} = (0, 0, \frac{k}{\rho}) \tag{1}$$

where $\rho = \sqrt{x^2 + y^2}$ and k is a non-vanishing constant. Its vector potential takes the expression

$$\mathbf{A} = \frac{k}{\rho}(-y, x, 0) \tag{2}$$

or $\mathbf{A} = k(-\sin\theta, \cos\theta, 0)$, in terms of the cylindrical coordinates (ρ, θ, z). Now, we want to describe the non-relativistic motion of an electron of mass m_0, and charge e subject to this magnetic potential. The corresponding Hamiltonian using cylindrical coordinates has the following form

$$H = \frac{P_\rho^2}{2m_0} + \frac{(P_\theta - \frac{ek}{c}\rho)^2}{2m_0\rho^2} + \frac{P_z^2}{2m_0} \tag{3}$$

where we recall the expressions of the canonical momenta

$$\dot\rho = \frac{P_\rho}{m_0}, \qquad \dot\theta = \frac{P_\theta}{m_0\rho^2} - \frac{ek}{m_0 c\rho}, \qquad \dot z = \frac{P_z}{m_0}. \tag{4}$$

Since the coordinates θ and z are cyclic the corresponding momenta will be constants of motion: $P_\theta = \ell$ (the angular momentum around z) and $P_z = \mu$ (the linear momentum along z). According to (4), this means that the velocity $\dot z$ will be constant but $\dot\theta$ will depend on the motion of ρ. After replacing these constants we are left with an effective Hamiltonian for the remaining variable ρ,

$$\begin{aligned} H_{\rm eff}(\rho) &= \frac{P_\rho^2}{2m_0} + \frac{\ell^2}{2m_0\rho^2} - \frac{ek\ell}{m_0 c\rho} + \frac{e^2k^2}{2m_0c^2} + \frac{\mu^2}{2m_0} \\ &\equiv \frac{P_\rho^2}{2m_0} + V_{\rm eff}(\rho) \end{aligned} \tag{5}$$

where the product $ek\ell$ must be positive if we want the effective potential $V_{\rm eff}(\rho)$ to have a minimum and allow for bounded motions. A schematic plot of this potential for such a case can be seen in Fig. 1. A situation where $ek\ell < 0$ is represented in Fig. 2, where $e = k = 1$ and $\ell = -1, -2, -3$.

Thus, we have a classical system in a three dimensional space with three (independent) constants of motion: H, P_θ and P_z. This means that our system is integrable, but not superintegrable. The

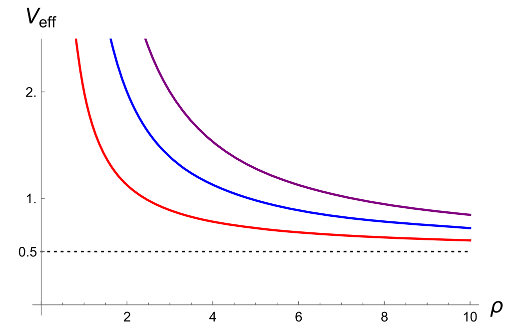

Fig. 2. Effective potential for $\ell = -1, -2, -3$ with the same values of the other parameters as in Fig. 1.

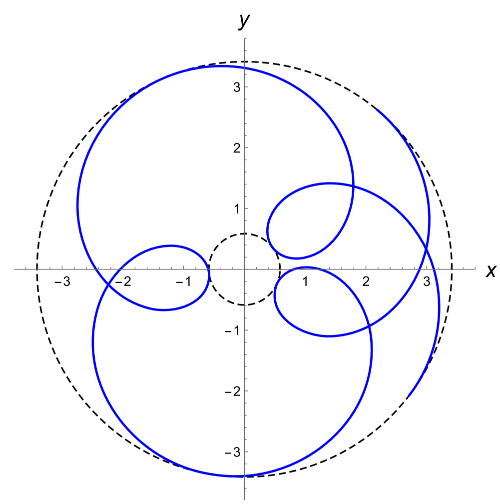

Fig. 3. Trajectory for the electron for $E = 0.25$ in continuous line, bounded by the inner and outer circles in dashing lines. The values of the parameters are $\ell = 1$, $\mu = 0$, $m_0 = 1$, $c = 1$, $k = 1$, $e = 1$.

equation of the projection of trajectory on the x–y plane for an energy E can readily be obtained from (4) and (5). If $\frac{\mu^2}{2m_0} \le E < \frac{e^2k^2}{2m_0c^2} + \frac{\mu^2}{2m_0}$ this trajectory is bounded and the equation for such orbits is

$$\begin{aligned} \theta(\rho) = &-\arcsin\left(\frac{\ell/\rho - ek/c}{\sqrt{2m_0E - \mu^2}}\right) \\ &- \frac{ek/c}{\sqrt{e^2k^2/c^2 + \mu^2 - 2m_0E}} \times \\ &\arcsin\frac{(e^2k^2/c^2 + \mu^2 - 2m_0E)\rho - \ell ek/c}{\ell\sqrt{2m_0E - \mu^2}}. \end{aligned} \tag{6}$$

A graphic of this type of bounded trajectories on the x–y plane is shown in Fig. 3. When $E > \frac{e^2k^2}{2m_0c^2} + \frac{\mu^2}{2m_0}$, the motion is unbounded and it is given by

$$\begin{aligned} \theta(\rho) = &-\arcsin\left(\frac{\ell/\rho - ek/c}{\sqrt{2m_0E - \mu^2}}\right) \\ &- \frac{ek/c}{\sqrt{2m_0E - e^2k^2/c^2 - \mu^2}} \times \\ &\operatorname{arccosh}\frac{(2m_0E - e^2k^2/c^2 - \mu^2)\rho + \ell ek/c}{\ell\sqrt{2m_0E - \mu^2}}. \end{aligned} \tag{7}$$

In Fig. 4 it is given the aspect of an unbounded trajectory. For both cases, the motion in the z-direction is uniform. The implicit time