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Integrability of Hamiltonian systems with algebraic potentials

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ABSTRACT

Problem of integrability for Hamiltonian systems with potentials that are algebraic thus multivalued functions of coordinates is discussed. Introducing potential as a new variable the original Hamiltonian system on $2n$ dimensional phase space is extended to $2n+1$ dimensional system with rational right-hand sides. For extended system its non-canonical degenerated Poisson structure of constant rank $2n$ and rational Hamiltonian is identified. For algebraic homogeneous potentials of non-zero rational homogeneity degree necessary integrability conditions are formulated. These conditions are deduced from an analysis of the differential Galois group of variational equations around particular solutions of a straight line type. Obtained integrability obstructions are applied to the class of monomial homogeneous potentials. Some integrable potentials satisfying these conditions are found.

1. Introduction

In this paper we consider natural Hamiltonian systems with n degrees of freedom for which Hamilton function has the form

$$H(\boldsymbol{q},\boldsymbol{p})=\frac{1}{2}\sum_{i=1}^{n}p_i^2+V(\boldsymbol{q}), \tag{1.1}$$

where $V(\boldsymbol{q})$ is a potential; $\boldsymbol{q}=(q_1,\dots,q_n)$ and $\boldsymbol{p}=(p_1,\dots,p_n)$ are canonical coordinates and momenta, respectively. Equations of motion read

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q}=\boldsymbol{p},\qquad \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p}=-\partial_{\boldsymbol{q}}V(\boldsymbol{q}), \tag{1.2}$$

where

$$\partial_{\boldsymbol{q}}V(\boldsymbol{q})=\left(\frac{\partial V}{\partial q_1}(\boldsymbol{q}),\dots,\frac{\partial V}{\partial q_n}(\boldsymbol{q})\right)=(\partial_1V(\boldsymbol{q}),\dots,\partial_nV(\boldsymbol{q}))\,.$$

We assume that potential $V(\boldsymbol{q})$ is an algebraic function. That is, it satisfies the following equation

$$F(\boldsymbol{q},V(\boldsymbol{q})):=\sum_{i=0}^{m}f_i(\boldsymbol{q})V(\boldsymbol{q})^i=0, \tag{1.3}$$

where $f_i(\boldsymbol{q})$ are complex polynomials, and $m>0$. If $m=1$, then $V(\boldsymbol{q})$ is a rational function

$$V(\boldsymbol{q})=-\frac{f_0(\boldsymbol{q})}{f_1(\boldsymbol{q})}. \tag{1.4}$$

If $m>1$, then, for a generic value of $\boldsymbol{q}\in\mathbb{C}^n$, equation (1.3) defines m different values of $V(\boldsymbol{q})$, so one has to fix a 'branch' of multi-valued function $V(\boldsymbol{q})$. For example, if $m=2$, then there are two branches

$$V(\boldsymbol{q})=\frac{-f_1(\boldsymbol{q})\pm\sqrt{f_1(\boldsymbol{q})^2-4f_2(\boldsymbol{q})f_0(\boldsymbol{q})}}{2f_2(\boldsymbol{q})}, \tag{1.5}$$

and a choice of sign before the radical fixes the branch. If $m>4$, then by fundamental theorem of algebra, generally it is impossible to express $V(\boldsymbol{q})$ in terms of radicals.

The fundamental question concerning investigated systems is their integrability. It seems that the most powerful methods for a study of the integrability are based on investigation of variational equations along a particular solution. Here we have in mind Ziglin and Morales–Ramis theories, see [19,20] and [13]. As it was noticed by Combot [3], these theories cannot be applied directly for investigation of systems with algebraic potentials because they were formulated for systems with complex meromorphic right-hand sides.

A general framework for study systems with potentials which are meromorphic functions on certain algebraic varieties in $\mathbb{C}^{n+s}$ was proposed by Combot [3]. Moreover, in this paper necessary conditions for integrability of algebraic homogeneous potentials of degree $k\in\mathbb{Z}^{\times}:=\mathbb{Z}\setminus\{0\}$ were derived. They coincide with the respective conditions obtained for meromorphic homogeneous potentials obtained by Morales-Ruiz and Ramis [14].

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The aim of this paper is twofold. At first we show that systems with algebraic potentials can be considered as restrictions of certain Poisson systems with rational Hamiltonians and rational Poisson tensors. The rank of these Poisson structures is constant and equal to $2n$.

Our second aim is to supplement consideration of Combot [3] for a homogeneous potential of an arbitrary degree $k\in\mathbb{Q}^{\times}$. In fact, knowing how to investigate algebraic potentials, the restriction of their degree of homogeneity to integer k is artificial, especially if we take into account that there are many examples of integrable and super-integrable systems with homogeneous potentials of non-integer degrees.

2. Algebraic potential

Classical examples of systems with algebraic potentials are Kepler problem

$$H=\frac{1}{2}\left(p_1^2+p_2^2+p_3^2\right)-\frac{\mu}{r},\qquad r^2=q_1^2+q_2^2+q_3^2, \tag{2.1}$$

or the generalised two fixed centers problem given by the following Hamiltonian

$$H=\frac{1}{2}\left(p_1^2+p_2^2+p_3^2\right)-\frac{\mu_1}{r_1^q}-\frac{\mu_2}{r_2^q},\qquad q\in\mathbb{Q}, \tag{2.2}$$

where

$$r_i^2=(q_1-a_i)^2+q_2^2+q_3^2,\qquad i=1,2, \tag{2.3}$$

see [12]. In both cases potential is expressed explicitly in terms of radicals, however in general case we have to consider $V(\boldsymbol{q})$ as an implicit function defined by (1.3). Thus, the right-hand sides of equations (1.2) are given implicitly. By the implicit function theorem, the partial derivatives of $V(\boldsymbol{q})$ can be expressed in terms of partial derivatives of polynomial

$$F(\boldsymbol{q},u):=\sum_{i=0}^{m} f_i(\boldsymbol{q})u^i\in\mathbb{C}[\boldsymbol{q},u]. \tag{2.4}$$

Namely, we can assume that F is irreducible and consider a point $(\boldsymbol{q}_0,u_0)\in\mathbb{C}^n\times\mathbb{C}$ such that $F(\boldsymbol{q}_0,u_0)=0$ and $\partial_u F(\boldsymbol{q}_0,u_0)\neq 0$. Then, in a neighbourhood $U\subset\mathbb{C}^n$ of $\boldsymbol{q}_0$, a function $U\ni\boldsymbol{q}\mapsto V(\boldsymbol{q})$ is well defined, and

$$\frac{\partial V}{\partial q_i}(\boldsymbol{q})=-\left[\frac{\partial F}{\partial u}(\boldsymbol{q},V(\boldsymbol{q}))\right]^{-1}\frac{\partial F}{\partial q_i}(\boldsymbol{q},V(\boldsymbol{q})), \tag{2.5}$$

for $i=1,\dots,n$. Notice that the right-hand side of the above formula is still implicit because we do not know how to evaluate $V(\boldsymbol{q})$, except for $\boldsymbol{q}_0$ for which we know that $V(\boldsymbol{q}_0)=u_0$. However, if $\boldsymbol{q}(t)$ evolves in time according to (1.2) and satisfies initial condition $\boldsymbol{q}(0)=\boldsymbol{q}_0$, then $u(t):=V(\boldsymbol{q}(t))$ fulfils initial condition $u(0)=V(\boldsymbol{q}_0)=u_0$. Moreover, evaluating the time derivative of $u(t)$ we find that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}u(t)&=\frac{\mathrm{d}}{\mathrm{d}t}V(\boldsymbol{q}(t))=\sum_{i=1}^{n}\frac{\partial V}{\partial q_i}(\boldsymbol{q}(t))\dot{q}_i(t)\\&=-\left[\frac{\partial F}{\partial u}(\boldsymbol{q}(t),u(t))\right]^{-1}\sum_{i=1}^{n}p_i(t)\frac{\partial F}{\partial q_i}(\boldsymbol{q}(t),u(t)).\end{aligned} \tag{2.6}$$

The above remark shows that instead of system (1.2) with implicit right hand sides it is better to consider the following system

$$\left.\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q}=\boldsymbol{p},\\&\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p}=\frac{1}{\partial_u F(\boldsymbol{q},u)}\partial_{\boldsymbol{q}}F(\boldsymbol{q},u),\\&\frac{\mathrm{d}}{\mathrm{d}t}u=-\frac{\boldsymbol{p}\cdot\partial_{\boldsymbol{q}}F(\boldsymbol{q},u)}{\partial_u F(\boldsymbol{q},u)},\end{aligned}\right\} \tag{2.7}$$

where we denoted

$$\partial_{\boldsymbol{q}}F=(\partial_1 F,\dots,\partial_n F),\qquad \partial_i F=\frac{\partial F}{\partial q_i},\quad\text{for}\quad i=1,\dots,n$$

and

$$\partial_u F(\boldsymbol{q},u)=\frac{\partial F}{\partial u}(\boldsymbol{q},u)=\sum_{i=0}^{m} i f_i(\boldsymbol{q})u^{i-1}.$$

For $\boldsymbol{a},\boldsymbol{b}\in\mathbb{C}^n$ we use notation

$$\boldsymbol{a}\cdot\boldsymbol{b}:=\sum_{i=1}^{n}a_ib_i.$$

The above construction is summarised in the following proposition.

Proposition 2.1. *Let $(\boldsymbol{q}(t),\boldsymbol{p}(t))$ be a solution of system (1.2) with initial condition $(\boldsymbol{q}(0),\boldsymbol{p}(0))=(\boldsymbol{q}_0,\boldsymbol{p}_0)$. Then $(\boldsymbol{q}(t),\boldsymbol{p}(t),u(t))$ with $u(t):=V(\boldsymbol{q}(t))$ is a solution of (2.7) satisfying initial condition $(\boldsymbol{q}(0),\boldsymbol{p}(0),u(0))=(\boldsymbol{q}_0,\boldsymbol{p}_0,V(\boldsymbol{q}_0))$.*

Moreover, system (2.7) has several interesting properties. At first we notice that it has two global first integrals.

Proposition 2.2. *System (2.7) has two polynomial first integrals*

$$K=\frac{1}{2}\boldsymbol{p}\cdot\boldsymbol{p}+u,\quad\textit{and}\quad F=F(\boldsymbol{q},u). \tag{2.8}$$

Proof of this proposition consists in a direct check and it is left to the reader.

Now our aim is to show that system (2.7) is Hamiltonian with respect to a certain non-canonical degenerated Poisson bracket defined in $\mathbb{C}^{2n+1}$. To this end, let us denote $\boldsymbol{z}=(\boldsymbol{q},\boldsymbol{p},u)\in\mathbb{C}^{2n+1}$. For two smooth functions $G(\boldsymbol{z})$ and $R(\boldsymbol{z})$ we define their bracket $\{G,R\}(\boldsymbol{z})$ in the following way

$$\begin{aligned}\{G,R\}&=\partial_{\boldsymbol{q}}G\cdot\partial_{\boldsymbol{p}}R-\partial_{\boldsymbol{p}}G\cdot\partial_{\boldsymbol{q}}R+\partial_u R(\boldsymbol{C}\cdot\partial_{\boldsymbol{p}}G)\\&\quad-\partial_u G(\boldsymbol{C}\cdot\partial_{\boldsymbol{p}}R),\end{aligned} \tag{2.9}$$

where

$$\boldsymbol{C}=\boldsymbol{C}(\boldsymbol{z}):=\frac{1}{\partial_u F}\partial_{\boldsymbol{q}}F. \tag{2.10}$$

Lemma 2.1. *The bracket defined by (2.9) is antisymmetric and satisfies the Leibniz and the Jacobi identities.*

Proof. The first two properties are satisfied in an obvious way. In order to prove that the bracket satisfies the Jacobi identity we notice that $\{z_i,z_j\}$ do not vanish only in the following cases

$$\{q_i,p_i\}=1,\qquad\{p_i,u\}=C_i\quad\text{for}\quad i=1,\dots,n. \tag{2.11}$$

Since $C_i=C_i(\boldsymbol{q},u)$, we also have

$$\{q_i,\{p_j,u\}\}=\{u,\{p_j,u\}\}=0\quad\text{for}\quad i,j=1,\dots,n. \tag{2.12}$$

In effect, the only non-vanishing triple brackets are the following ones