Physics Letters A 380 (2016) 142–146

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Maximal stochastic transport in the Lorenz equations

Sahil Agarwal [a], J.S. Wettlaufer [a,b,c,d,*]

[a] Program in Applied Mathematics, Yale University, New Haven, USA
[b] Departments of Geology & Geophysics, Mathematics and Physics, Yale University, New Haven, USA
[c] Mathematical Institute, University of Oxford, Oxford, UK
[d] Nordita, Royal Institute of Technology and Stockholm University, Stockholm, Sweden

ARTICLE INFO

Article history:
Received 15 August 2015
Accepted 28 September 2015
Available online 3 October 2015
Communicated by F. Porcelli

Keywords:
Lorenz equations
Rayleigh–Bénard convection
Heat transport
Stochastic dynamics
Upper bounds

ABSTRACT

We calculate the stochastic upper bounds for the Lorenz equations using an extension of the background method. In analogy with Rayleigh–Bénard convection the upper bounds are for heat transport versus Rayleigh number. As might be expected, the stochastic upper bounds are larger than the deterministic counterpart of Souza and Doering [1], but their variation with noise amplitude exhibits interesting behavior. Below the transition to chaotic dynamics the upper bounds increase monotonically with noise amplitude. However, in the chaotic regime this monotonicity depends on the number of realizations in the ensemble; at a particular Rayleigh number the bound may increase or decrease with noise amplitude. The origin of this behavior is the coupling between the noise and unstable periodic orbits, the degree of which depends on the degree to which the ensemble represents the ergodic set. This is confirmed by examining the close returns plots of the full solutions to the stochastic equations and the numerical convergence of the noise correlations. The numerical convergence of both the ensemble and time averages of the noise correlations is sufficiently slow that it is the limiting aspect of the realization of these bounds. Finally, we note that the full solutions of the stochastic equations demonstrate that the effect of noise is equivalent to the effect of chaos.

1. Introduction

Noise is an integral part of any physical system. It can be ascribed to fluctuations arising from intermittent forcing, observational uncertainties, interference from external sources or unresolved physics. In circumstances where noise acts to destroy a signal of interest, it is viewed as a nuisance. However, it can also be the case that fluctuations act to stabilize a system, examples of which include noise-induced optical multi-stability [2], asymmetric double well potentials [3], plant ecosystems [4], population dynamics [5], and in electron–electron interactions in quantum systems [6]. Curiously, it has recently been shown that noise can have positive effects on cognitive functions such as learning and memory [7]. Finally, a key issue arising when examining observational data is whether fluctuations are intrinsic or due to external forcing, which can be confounded by temporal multifractality (e.g. [8]).

Given the breadth of settings in which the effects of noise manifest themselves on dynamical systems, it appears prudent to examine such matters in a well studied and yet broadly relevant system. Thus, we study the influence of noise in the Lorenz system [9], which is an archetype of deterministic nonlinear dynamics. Moreover, Souza and Doering [1] have recently determined the maximal (upper bounds) transport in the Lorenz equations, thereby providing us with a rigorous test bed for stochastic extensions. In Section 2 we describe the stochastic Lorenz model, followed by the derivation of the stochastic upper bounds in Section 3. We interpret the core results and their implications in Section 4 before concluding.

2. Stochastic Lorenz model

The Lorenz model is a Galerkin-modal truncation of the equations for Rayleigh–Bénard convection with stress-free boundary conditions on the upper and lower boundaries. It acts as a rich toy model of low-dimensional chaos and since its origin extensive studies have been made spanning a wide range of areas (e.g. [10]).

* Corresponding author at: Yale University, 210 Whitney Ave., P.O. Box 208109, New Haven, CT, 06520-8109, USA.
E-mail addresses: sahil.agarwal@yale.edu (S. Agarwal), john.wettlaufer@yale.edu (J.S. Wettlaufer).

http://dx.doi.org/10.1016/j.physleta.2015.09.046
0375-9601/

Of particular relevance here, is using the system as a model for heat transport in high Rayleigh number turbulent convection [1].

The stochastic form of the Lorenz system is described by the following coupled nonlinear ordinary differential equations,

$$\frac{d}{dt}X = \sigma(Y - X) + A_1\xi_1,$$
$$\frac{d}{dt}Y = X(\rho - Z) - Y + A_2\xi_2,$$
$$\frac{d}{dt}Z = XY - \beta Z + A_3\xi_3 \quad (1)$$

where X describes the intensity of convective motion, Y is the temperature difference between ascending and descending flow and Z is the deviation from linearity of the vertical temperature profile. The control parameters are σ the *Prandtl Number*, ρ the *Rayleigh Number* and β a domain geometric factor. The A_i are the noise amplitudes and ξ_i are the noise processes. Clearly, the deterministic system has $A_i = 0$.

This type of additive noise may appear, for example, in observational errors, when the errors do not depend on the system state or as a model of sub-grid scale processes approximated by noise associated with unexplained physics [11]. In multiplicative noise the system has an explicitly state dependent noise process.

Although real noise will always have a finite time correlation, taking the limit that the noise correlation goes to zero as $\Delta t \to 0$, serves as a good approximation for the noise forcing. This is the *white noise* limit of colored noise forcing. White noise forcing $\xi(t)$ is defined by an autocorrelation function written as

$$\langle \xi(t)\xi(s)\rangle = 2D\delta(t - s), \quad (2)$$

where, $t - s$ is the time lag, D is the amplitude of the noise, $\langle\bullet\rangle$ represents the time average and $\delta(r)$ is the Dirac delta-function.

3. Stochastic maximal transport

Initiated by the work of Louis Howard [12], maximizing the transport of a quantity such as heat or mass is a core organizing principle in modern studies of dissipative systems. In this spirit Souza and Doering [1] studied the transport in the deterministic Lorenz equations and determined the upper bound, which depends on the exact steady solutions X_s, Y_s, as $\overline{\lim}_{T\to\infty} \langle XY\rangle_T = X_sY_s = \beta(\rho - 1)$, where $X_s = Y_s = \pm\sqrt{\beta(\rho - 1)}$ for $\rho \geq 1$. Moreover, they showed that any time-dependent forcing would decrease the transport in the system, and hence the steady state maximizes the transport in the system. We study the effect of noise on the maximal transport in this system as the Rayleigh number ρ is varied, with $\sigma = 10$ and $\beta = 8/3$ fixed.

Let $X = x$, $Y = \rho y$, $Z = \rho z$ and $A_1 = A_2 = A_3 = A$ in the system of equations (1), which transform to

$$\frac{d}{dt}x = \sigma(\rho y - x) + A\xi_1,$$
$$\frac{d}{dt}y = x(1 - z) - y + \frac{A}{\rho}\xi_2,$$
$$\frac{d}{dt}z = xy - \beta z + \frac{A}{\rho}\xi_3. \quad (3)$$

In the next two sub-sections, we calculate the stochastic upper bound of equations (3) using both *Itô* and *Stratonovich* calculi.

3.1. Itô calculus framework

Now, knowing that the state variables (x, y, z) in the Lorenz system are bounded [1,13], and following the approach of Souza and Doering [1] for this stochastic system, the long time averages of $\frac{1}{2}x^2$, $\frac{1}{2}(y^2 + z^2)$ and $-z$ can be written as

$$0 = -\langle x^2\rangle_T + \rho\langle xy\rangle_T + \frac{A^2}{2\sigma} + \frac{A}{\sigma}\langle x\xi_1\rangle_T + O(T_{-1}), \quad (4)$$

$$0 = -\langle y^2\rangle_T + \langle xy\rangle_T - \beta\langle z^2\rangle_T + \frac{A^2}{\rho^2} + \frac{A}{\rho}\langle y\xi_2\rangle_T + \frac{A}{\rho}\langle z\xi_3\rangle_T + O(T_{-1}), \quad (5)$$

$$0 = -\langle xy\rangle_T + \beta\langle z\rangle_T + O(T_{-1}), \quad (6)$$

where the terms $\frac{A^2}{2\sigma}$ in Eq. (4) and $\frac{A^2}{\rho^2}$ in Eq. (5) are a consequence of *Itô's lemma.*

Now, let $z = z_0 + \lambda(t)$, where $z_0 = \frac{r-1}{r}$ is time-independent [1], and equations (5) and (6) now become,

$$0 = -\langle y^2\rangle_T + \langle xy\rangle_T - \beta z_0^2 - 2\beta z_0\langle\lambda\rangle_t - \beta\langle\lambda^2\rangle_T + \frac{A^2}{\rho^2} + \frac{A}{\rho}\langle y\xi_2\rangle_T + \frac{A}{\rho}\langle\lambda\xi_3\rangle_T + O(T_{-1}), \text{ and} \quad (7)$$

$$0 = -\langle xy\rangle_T + \beta z_0 + \beta\langle\lambda\rangle_T + O(T_{-1}). \quad (8)$$

Therefore, equation (7) + $2z_0$ × (8) becomes,

$$0 = -\langle y^2\rangle_T + (1 - 2z_0)\langle xy\rangle_T + \beta z_0^2 - \beta\langle\lambda^2\rangle_T + \frac{A^2}{\rho^2} + \frac{A}{\rho}\langle y\xi_2\rangle_T + \frac{A}{\rho}\langle\lambda\xi_3\rangle_T + O(T_{-1}). \quad (9)$$

Now adding $\frac{1}{\rho}$ × (4) to ρ × (9) gives

$$0 = -\rho\langle y^2\rangle_T + \rho(1 - 2z_0)\langle xy\rangle_T + \rho\beta z_0^2 - \rho\beta\langle\lambda^2\rangle_T - \frac{1}{\rho}\langle x^2\rangle_T + \langle xy\rangle_T + \frac{A}{\rho\sigma}\langle x\xi_1\rangle_T + \frac{A^2}{\rho} + \frac{A^2}{2\rho\sigma} + A\langle y\xi_2\rangle_T + A\langle\lambda\xi_3\rangle_T + O(T_{-1}), \quad (10)$$

and adding $(\rho - 1)\langle xy\rangle_T$ to both sides gives,

$$(\rho - 1)\langle xy\rangle_T = \rho\beta z_0^2 + A\left[\langle y\xi_2\rangle_T + \langle\lambda\xi_3\rangle_T + \frac{1}{\sigma\rho}\langle x\xi_1\rangle_T\right] - \left\langle\left(\frac{x}{\sqrt{\rho}} - \sqrt{\rho}y\right)^2 + \rho\beta\lambda^2\right\rangle_T + A^2\left[\frac{1}{\rho} + \frac{1}{2\rho\sigma}\right] + O(T^{-1}). \quad (11)$$

We thus arrive at

$$(\rho - 1)\langle xy\rangle_T \leq \rho\beta z_0^2 + A^2\left[\frac{1}{\rho} + \frac{1}{2\rho\sigma}\right] + A\left[\langle y\xi_2\rangle_T + \langle\lambda\xi_3\rangle_T + \frac{1}{\sigma\rho}\langle x\xi_1\rangle_T\right] + O(T^{-1}). \quad (12)$$

Comparing equation (12) above with equation 19 from Souza and Doering [1], we see two additional terms;

$$\overline{\lim_{T\to\infty}}\langle XY\rangle_T = \overline{\lim_{T\to\infty}}\rho\langle xy\rangle_T \leq \beta(\rho - 1) + \frac{A^2}{\rho - 1}\left[1 + \frac{1}{2\sigma}\right]$$