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Pattern formation in annular systems of repulsive particles

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ABSTRACT

General particle models with symmetric and asymmetric repulsion are studied and investigated for finite-range and exponential interaction in an annulus. In the symmetric case transitions from one- to multi-lane behavior including multistability are observed for varying particle density and for a varying curvature with fixed density. Hence, the system cannot be approximated by a periodic channel. In the asymmetric case, which is important in pedestrian dynamics, we reveal an inhomogeneous new phase, a traveling wave reminiscent of peristaltic motion.

1. Introduction

Many physical systems can be described by particle models. The interaction between these particles is often modeled by forces, which typically depend on the inter-particle distance, e.g., gravitational attraction in celestial mechanics [1], Coulomb forces between charged particles [2] or swarming models of self-propelled particles [3]. In most physical systems Newton's third law of *actio–reactio* is valid. However, when considering a larger class of interacting particle models, it might be crucial to introduce an asymmetry into the interaction terms, such that the forces not only depend on the distance, but also on direction. Examples are found in pedestrian models, where pedestrians typically pay more attention to people in front than behind [4,5], or in traffic dynamics, where drivers on highways are assumed to adjust their speed according to the distance to the following car [6–8]. In order to isolate fundamental effects, experiments are often conducted in simple geometries such as an annulus or a torus. These include the tokamak [9], the large hadron collider [10], camphor boats in a circular duct [11], pedestrian experiments in annular geometry [12] or traffic studies on a ring road [13].

Motivated by traffic and pedestrian models, it seems valuable to study particle systems with asymmetric interaction in an annular geometry where Newton's third law is invalid. An interesting aspect of these systems is pattern formation. Recently, the stability of zigzag patterns in a channel with repulsive particles has been studied in [14] for different interaction potentials. Depending on parameter values, three fundamentally different solutions have been found: one-lane flow, homogeneous two-lane flow and inhomogeneous two-lane flow (denoted *distorted zigzag* in [14]). Furthermore, the transition to collective motion like flocking and multivortex dynamics depending on density and boundary conditions has been analyzed for self-propelled particle models in [15,16]. Experimental verification for desert locusts, forming marching bands, has been given in [17].

Here we study a model of interacting point particles with finite-range [18] and exponential interaction [19] in an annular geometry. In particular, we focus on the resulting patterns that emerge from the dynamics of the model. For certain parameter values, coexistence of patterns depending on the initial state is observed. Further, we introduce asymmetry into the model and investigate the influence on the possible patterns.

2. Particle model with asymmetric interaction

In order to study the dynamics of particles confined in an annulus, we shortly review the models of [18,19] where pedestrians are modeled as interacting particles without extension and generalize them to include asymmetry. The equations of motion are described by forces that act on the particles (so-called *social forces*

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in the context of pedestrian dynamics). The position $\boldsymbol{r}_i \in \mathbb{R}^2$ of particle $i \in \{1, \ldots, N\}$ evolves according to

$$\ddot{\boldsymbol{r}}_i = \boldsymbol{F}_i + \sum_{w \in W} \boldsymbol{F}_{iw}^{W} + \sum_{j \neq i} \boldsymbol{F}_{ij}^{P}, \tag{1}$$

where $\boldsymbol{F}_i$ is its target or external force and $\boldsymbol{F}_{iw}^{W}$ and $\boldsymbol{F}_{ij}^{P}$ describe the interaction with walls and other particles, respectively. In many pedestrian and traffic models the target force is given as

$$\boldsymbol{F}_i = \tau^{-1}\left(v_0 \boldsymbol{e}_i - \boldsymbol{v}_i\right) \tag{2}$$

with reaction time τ, desired speed v_0 and velocity of pedestrian $\boldsymbol{v}_i = \dot{\boldsymbol{r}}_i$. The desired direction vector $\boldsymbol{e}_i$ is a normalized vector pointing in the direction of the target. For general particle models, this contribution might be replaced by an external driving force instead of an internal desire to reach a target.

The interaction with walls ($\nu = W$) and other particles ($\nu = P$) is determined by

$$\boldsymbol{F}_{i\eta}^{\nu} = f^{\nu}(\|\boldsymbol{r}_{i\eta}\|)\frac{\boldsymbol{r}_{i\eta}}{\|\boldsymbol{r}_{i\eta}\|}, \tag{3}$$

where η indexes either walls ($\eta = w$) or other particles ($\eta = j \neq i$). The distance vector $\boldsymbol{r}_{iw}$ points from the wall w to particle i in the normal direction and $\boldsymbol{r}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j$ is the vector from particle j to i. We investigate two types of interaction forces f^{ν} (cf. Fig. 1(b)): a finite-range interaction [20]

$$\hat{f}_{\hat{A},\hat{B}}(x) = -\hat{A}\left[\tan\left(\alpha_{\hat{B}}(x)\right) - \alpha_{\hat{B}}(x)\right] H(\hat{B} - x) \tag{4}$$

with Heaviside step function H, $\alpha_{\hat{B}}(x) = \frac{\pi}{2}\left(\frac{x}{\hat{B}} - 1\right)$ and an exponential interaction

$$\tilde{f}_{\tilde{A},\tilde{B}}(x) = \tilde{A}\exp(-\tilde{B}x). \tag{5}$$

For a smooth switching between (4) and (5), we use

$$f^{\nu}(x) = \lambda \hat{f}_{\hat{A}^{\nu},\hat{B}^{\nu}}(x) + (1 - \lambda)\tilde{f}_{\tilde{A}^{\nu},\tilde{B}^{\nu}}(x), \tag{6}$$

where $\lambda \in [0, 1]$ is a homotopy parameter. The constants A^{ν} and B^{ν} in hat ˆ and tilde ˜ notation describe the interaction with walls and particles in the finite-range and the exponential model, respectively.

Inspired by an asymmetric attention of pedestrians to their close surroundings we propose a modification of (3). Pedestrians are assumed to adjust to the average velocity of their neighborhood given by

$$\boldsymbol{V}_i = \sum_{j=1}^{N} \boldsymbol{v}_j H(\delta - \|\boldsymbol{r}_{ij}\|), \tag{7}$$

with local radius of interaction δ. After normalization we obtain the average walking direction of the neighbors as

$$\boldsymbol{n}_i = \frac{\boldsymbol{V}_i}{\|\boldsymbol{V}_i\|}. \tag{8}$$

In order to incorporate this information into the model, we modify (3) for the particle interaction such that

$$\boldsymbol{F}_{ij}^{P} = \left[1 + \varepsilon \frac{\boldsymbol{r}_{ij}}{\|\boldsymbol{r}_{ij}\|} \cdot \boldsymbol{n}_i\right] f^{P}(\|\boldsymbol{r}_{ij}\|)\frac{\boldsymbol{r}_{ij}}{\|\boldsymbol{r}_{ij}\|}, \tag{9}$$

where ε is an asymmetry parameter that describes the deviation from the original problem. Note that for $\varepsilon \neq 0$, Newton's third law is violated and the interaction force depends on the relation between the difference vector $\boldsymbol{r}_{ij}$ and the velocity direction of the neighborhood $\boldsymbol{n}_i$. In general particle models the importance of the neighborhood could appear in setups with shielding effects, see [21], e.g.

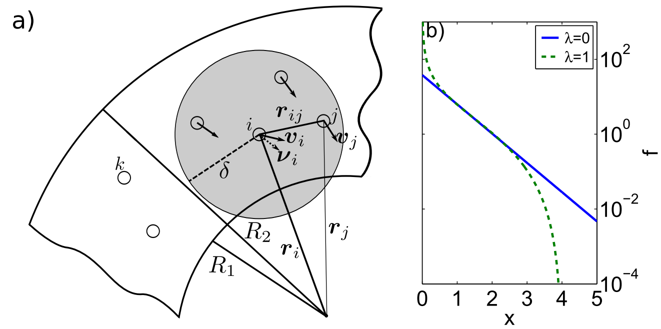

Fig. 1. (a) Sketch of the annular geometry with inner and outer radius R_1 and R_2, respectively. The vectors $\boldsymbol{r}_i$, $\boldsymbol{r}_j$ and $\boldsymbol{r}_{ij}$ describe the positions and the connecting vector of particles i and j. The δ-neighborhood of particle i is denoted by the circular region, wherein particles contribute to the average walking direction $\boldsymbol{n}_i$ with their respective velocities $\boldsymbol{v}_j$. (b) Logarithmic plot of particle interaction forces (4) and (5) as used in the simulations. Eq. (4) has a finite range $\hat{B}$ and diverges at 0, while (5) has infinite range and is finite at 0.

3. Results

We study the behavior of the model (1)–(9) in an annular geometry. The following set of (dimensionless) parameter values is used as an example throughout the paper: $\tau = 0.22$, $v_0 = 2.5$, $\hat{A}^P = 5$, $\hat{B}^W = 3$, $\hat{B}^P = 4$, $\tilde{A}^W = 115$, $\tilde{A}^P = 38$, $\tilde{B}^W = 2.3$, $\tilde{B}^P = 1.8$, $\delta = 2$. We specify varying parameters $N, \varepsilon, \lambda, R_1, R_2$ and $\hat{A}^W$ for each investigation separately. Simulations are performed using a Runge–Kutta scheme implemented in MATLAB's [22] `ode45` integrator.

3.1. Geometry

Motivated by the examples in the Introduction, we define the geometry of our problem as an annulus with inner radius R_1 and outer radius R_2. The direction vector $\boldsymbol{e}_i$ for particle i is chosen such that it moves clockwise around the annulus. A sketch of a sector of the geometry with particles drawn as small open circles is shown in Fig. 1(a). The shaded region defines the δ-neighborhood of particle i and $\boldsymbol{v}_i$ is the average walking direction of the neighborhood.

3.2. Symmetric interaction $\varepsilon = 0$

We first study the symmetric case with $\varepsilon = 0$ for the finite-range particle model $\lambda = 1$ and fix the number of particles $N = 160$. In [23,24] the density was shown to be of importance for the resulting stationary states. We vary the particle density in a smooth way by keeping the particle number fixed and using different values of the outer ring radius $R_2 \in [4, 35]$ while the width of the corridor is kept fixed, i.e., $R_1 = R_2 - 3.5$. The lane formation behavior can be understood by investigating the radial distance between particle lanes

$$b = \max_i \|\boldsymbol{r}_i\| - \min_i \|\boldsymbol{r}_i\|. \tag{10}$$

For a simple one-lane state, b is zero, while for multi-lane states, b assumes positive values.

The results of a simulation sweep varying R_2 are shown in Fig. 2(a). After a parameter change the system is integrated for 300 time steps in order to converge to a stationary state. For large values of R_2, corresponding to small particle density, a one-lane state is obtained. By subsequently decreasing R_2 in a downsweep, cascading transitions to multi-lane states are observed. The transitions resemble the typical square-root behavior of a pitch-fork bifurcation. A following upsweep for increasing R_2 reveals small deviations in b compared to the downsweep data. This suggests, that the system exhibits multistability and that the stationary state