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Derivation of the Camassa–Holm equations for elastic waves

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ABSTRACT

In this paper we provide a formal derivation of both the Camassa–Holm equation and the fractional Camassa–Holm equation for the propagation of small-but-finite amplitude long waves in a nonlocally and nonlinearly elastic medium. We first show that the equation of motion for the nonlocally and nonlinearly elastic medium reduces to the improved Boussinesq equation for a particular choice of the kernel function appearing in the integral-type constitutive relation. We then derive the Camassa–Holm equation from the improved Boussinesq equation using an asymptotic expansion valid as nonlinearity and dispersion parameters that tend to zero independently. Our approach follows mainly the standard techniques used widely in the literature to derive the Camassa–Holm equation for shallow-water waves. The case where the Fourier transform of the kernel function has fractional powers is also considered and the fractional Camassa–Holm equation is derived using the asymptotic expansion technique.

1. Introduction

In the present paper we show that, in the long wave limit, small-but-finite waves propagating in a one-dimensional medium made of nonlocally and nonlinearly elastic material satisfy the Camassa–Holm (CH) equation [1] and the fractional CH equation (see Eq. (4.10)) when a proper balance between dispersion and nonlinearity exists.

The CH equation

$$v_\tau+\kappa_1 v_\zeta+3vv_\zeta-v_{\zeta\zeta\tau}=\kappa_2(2v_\zeta v_{\zeta\zeta}+vv_{\zeta\zeta\zeta}), \tag{1.1}$$

was derived for the propagation of unidirectional small-amplitude shallow-water waves [1–6] when the nonlinear effects are stronger than the dispersive effects. Due to the fact that, even for smooth initial data, the solution of the CH equation stays bounded as its slope becomes unbounded, it is often used as an appropriate model capturing the essential features of wave-breaking of shallow-water waves [7]. However, recalling that (1.1) is derived under the long wavelength assumption, it follows that the CH equation is valid only when the solutions and their derivatives remain bounded [5]. For a discussion on a different criterion for wave-breaking in long wave models we refer the reader to [8]. It is interesting to note that the CH equation as a model for wave-breaking of water waves is an infinite-dimensional completely integrable Hamiltonian system [9,10]. Another interesting property of the CH equation is the existence of the so-called peakon solitary wave solutions when $\kappa_1=0$ [1]. At this point, it is worth pointing out that the derivation in the present study is also based on the long wavelength assumption and that κ_1 is nonzero for the resulting equation. In addition to the studies about water waves, there are also studies that derive the CH equation as an appropriate model equation for nonlinear dispersive elastic waves. We refer the reader to [11] for the derivation of a CH-type equation governing the propagation of long waves in a compressible hyperelastic rod, and to [12] for the derivation of a two-dimensional CH-type equation governing the propagation of long waves in a compressible hyperelastic plate. However, these studies relied only on the "geometrical" dispersion resulting from the existence of the boundaries, that is, from the existence of a bounded elastic solid, like a rod or a plate. Another type of dispersion for elastic waves is the "physical" dispersion produced by the internal structure of the medium. Therefore, one interesting question is to investigate whether the CH equation can be derived as an asymptotic approximation for physically dispersive nonlinear elastic waves in the absence of the geometrical dispersion. In this study, we consider the one-dimensional wave propagation in an infinite, nonlinearly and nonlocally elastic medium whose constitutive behavior is described by a convolution integral. We then show that, for an exponential-type kernel function, the CH equation can model the propagation of elastic waves even in the absence of the geometrical dispersion. Furthermore, by considering a fractional-type kernel function we are able to derive a fractional-type CH equation, which indicates the possibility of obtaining more general evolution equa-

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tions for suitable kernel functions. It is well known that the KdV and BBM equations are valid at the same level of approximation while the CH equation is more accurate than the KdV and BBM equations. Therefore, when we neglect the highest order terms in the asymptotic expansion, the KdV and BBM equations and their fractional generalizations are also obtained as a by-product of the present derivation. We underline that the asymptotic derivation of the CH equation needs a double asymptotic expansion in two small parameters characterizing nonlinear and dispersive effects. However, assuming simply that the two parameters are equal, the asymptotic derivations of the KdV and BBM equations can also be based on a single asymptotic expansion in one small parameter resulting from the balance of nonlinear and dispersive effects.

The paper is organized as follows. Section 2 presents the governing equations of one-dimensional nonlocal nonlinear elasticity theory and gives the equation of motion in dimensionless quantities for various forms of the kernel function. In Section 3, using a multiple scale asymptotic expansion, the CH equation is derived from the improved Boussinesq (IBq) equation which is the equation of motion for the exponential kernel function. Section 4 presents the derivation of a fractional CH equation from the equation of motion corresponding to a fractional-type kernel function.

2. A one-dimensional nonlinear theory of nonlocal elasticity

We consider a one-dimensional, infinite, homogeneous, elastic medium with a nonlinear and nonlocal stress–strain relation (see [13–15] and the references cited therein for a more detailed discussion of the nonlocal model). In the absence of body forces the equation of motion is

$$\rho_0 u_{tt} = \big(S(u_X)\big)_X, \tag{2.1}$$

where the scalar function $u(X,t)$ represents the displacement of a reference point X at time t, ρ_0 is the mass density of the medium, $S = S(u_X)$ is the stress and the subscripts denote partial derivatives. In contrast with classical elasticity, we take the constitutive equation for the stress S as a general nonlinear and nonlocal function of the strain u_X. That is, we assume that the stress at a reference point is a nonlinear function of the strain at all points in the body. As in [14,15], the constitutive equation of the present model has the form

$$S(X,t) = \int_{\mathbb{R}} \alpha\big(|X-Y|\big)\sigma(Y,t)\,\mathrm{d}Y,$$
$$\sigma(X,t) = W'\big(u_X(X,t)\big) \tag{2.2}$$

where σ is the classical (local) stress, W is the strain-energy density function, Y denotes a generic point of the medium, α is a kernel function to be specified below, and the symbol $'$ denotes differentiation. The kernel α acts as a weight function that determines the relative contribution of the local stress $\sigma(Y,t)$ at a point Y in a neighborhood of X to the nonlocal stress $S(X,t)$. So, when the kernel becomes the Dirac delta function, the classical constitutive relation of a hyperelastic material is recovered. Assuming the reference configuration is a stress-free undistorted state, we require that $W(0) = W'(0) = 0$. We point out that if we take $W(u_X) = (\lambda+2\mu)(u_X)^2/2$ where λ and μ are Lame constants, the above equations reduce to those of the linear theory of one-dimensional nonlocal elasticity (see [13]).

Without loss of generality, for convenience, the strain-energy density function may be considered to consist of a quadratic part $(u_X)^2/2$ and a non-quadratic part $G(u_X)$ with $G(0) = G'(0) = 0$:

$$W(u_X) = \gamma\left[\frac{1}{2}(u_X)^2 + G(u_X)\right],$$

where γ is a constant with the dimension of stress. Differentiating both sides of (2.1) with respect to X and using (2.2) we obtain the equation of motion for the strain:

$$\rho_0 u_{Xtt} = \gamma\left\{\int_{\mathbb{R}} \alpha\big(|X-Y|\big)\big[u_X + g(u_X)\big]\mathrm{d}Y\right\}_{XX}, \tag{2.3}$$

where $g(s) = G'(s)$ with $g(0) = 0$. Now we define the dimensionless independent variables

$$x = \frac{X}{l}, \qquad \eta = \frac{t}{l}\sqrt{\frac{\gamma}{\rho_0}}$$

where l is a characteristic length and from now on, and for simplicity, we use u for u_X and t for η. Thus, (2.3) takes the form

$$u_{tt} = \big(\beta * \big(u + g(u)\big)\big)_{xx}, \tag{2.4}$$

where the convolution operator $*$ is defined by

$$\beta * v = \int_{\mathbb{R}} \beta(x-y)v(y)\,\mathrm{d}y$$

and $\beta(x) = l\alpha(|x|)$. From a wave propagation point of view, the harmonic wave solutions to the linearized form of (2.4) are dispersive and the sole source of dispersion in the present model is the internal structure of the medium but not the existence of the boundaries. In general the kernel function β is even, nonnegative and monotonically decreasing for $x > 0$ (we refer the reader to [13] for the properties that an admissible kernel function must satisfy). A list of the most commonly used kernel functions is given in [15]. Here we consider two kernel functions: the exponential kernel [16] which is the most widely used kernel function in the engineering applications of nonlocal elasticity [13,17], and a fractional-type kernel function. These two kernels are chosen because they are the simplest representatives of the kernels that are convenient for asymptotic expansions. Moreover, as discussed in Remark 4.2, starting from (2.4) with a general kernel satisfying some mild assumptions will lead to the same results with those of the two representative kernels.

The exponential kernel is given by $\beta(x) = \frac{1}{2}e^{-|x|}$. The Fourier transform of β is $\widehat{\beta}(\xi) = (1+\xi^2)^{-1}$ where ξ is the Fourier variable. Note that $\beta(x)$ is the Green's function for the operator $1 - D_x^2$ where D_x represents the partial derivative with respect to x. Now, using the convolution theorem stating that the Fourier transform of the convolution of two functions is the product of their Fourier transforms, we take the Fourier transform of both sides of (2.4). Then, substituting $\widehat{\beta}(\xi)$ of the exponential kernel into the resulting equation and taking the inverse Fourier transform, we obtain the equation of motion corresponding to the exponential kernel. Thus, for the exponential kernel, the equation of motion, (2.4), reduces to the IBq equation

$$u_{tt} - u_{xx} - u_{xxtt} = \big(g(u)\big)_{xx}. \tag{2.5}$$

We next consider a fractional-type kernel function whose Fourier transform is $\widehat{\beta}(\xi) = (1+(\xi^2)^{\nu})^{-1}$ where ν may not be an integer. Note that the previous case corresponds to $\nu = 1$. To ensure the local well-posedness of the Cauchy initial-value problem defined for the resulting form of the equation of motion [15], we impose the condition $\nu \geq 1$. Noting that $\beta(x)$ is the Green's function for the operator $1 + (-D_x^2)^{\nu}$, this time the equation of motion, (2.4), becomes an improved Boussinesq equation of fractional type

$$u_{tt} - u_{xx} + \big(-D_x^2\big)^{\nu} u_{tt} = \big(g(u)\big)_{xx}. \tag{2.6}$$

Here the operator $(-D_x^2)^{\nu}$ is defined as $(-D_x^2)^{\nu} q = \mathcal{F}^{-1}(|\xi|^{2\nu}\mathcal{F}q)$ where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse,