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# The gravitational time delay in the field of a slowly moving body with arbitrary multipoles

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ABSTRACT

We calculate the time delay of light in the gravitational field of a slowly moving body with arbitrary multipoles (mass and spin multipole moments) by the Time-Transfer-Function (TTF) formalism. The parameters we use, first introduced by Kopeikin for a gravitational source at rest, make the integration of the TTF very elegant and simple. Results completely coincide with expressions from the literature. The results for a moving body (with constant velocity) with complete multipole-structure are new, according to our knowledge.



## 1. Introduction

Light propagation in gravitational fields is a very important topic not only for modern astrometry because of the high accuracies achieved in modern observations, but also for other kinds of measurements such as radar ranging to spacecrafts or planets. Gravitational fields cause a propagation-time delay and a deflection of light-rays as well as a frequency shift of the involved photons. The first effect, called Shapiro delay [1], has to be considered in space-techniques such as Very-Long-Baseline Interferometry (VLBI), Lunar Laser Ranging (LLR), etc. The present VLBI model recommended by the IERS conventions 2010 [2] (the consensus model [3]) has an accuracy at the 1 picosecond level; it will be improved to 0.1 picosecond accuracy in the near future; LLR is approaching now the millimeter level [4]. The gravitational field of the Sun produces a maximum of about 100 nanoseconds for the Earth bounded VLBI observations [5] and 50 nanoseconds (15 meters) in LLR experiments.

The light propagation delay in the gravitational field of a stationary mass-monopole is quite easy to derive. For a body with arbitrary mass and spin multipole moments, moving with some velocity in the underlying coordinate system, the treatment becomes non-trivial. One usually uses the null geodesic equation to get the light-propagation information between two events (for example, emission and reception). The solutions for a gravitating body with arbitrary multipoles obtained in this way was first derived in Refs. [6,7]. Bertone et al. showed that the so-called Time-Transfer-Function (TTF) formalism can also be used efficiently to get the gravitational time-delay, but they dealt with the case of mass-monopoles only [8]. Recently, some authors discussed the light propagation in the field of a moving axisymmetric body [9].

In this Letter, we derive the TTF by means of special parameters and techniques that were first introduced by Kopeikin [6,7]; using this approach simplifies the calculations drastically. Results for the Shapiro-effect for a body with arbitrary mass- and spin-multipoles are obtained in a few lines. Our results completely coincide with the ones from the literature (e.g., [6,7]). This calculation is then generalized in a very simple way to the case of a body moving with slow and constant velocity in the underlying coordinate system.

In the next section, the metric of a body with arbitrary multipole-moments is recalled. In Sections 3 and 4, we introduce the TTF, and calculate the light propagation for the cases of arbitrary multiple moments and constant velocity. The last section contains conclusions and discussions.

## 2. The time transfer function

We will consider the propagation of light-signals in a first order post-Newtonian metric of form

$$g_{00} = -1 + \frac{2w}{c^2},$$

$$g_{0i} = -\frac{4}{c^3} w_i,$$

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$$g_{ij} = \delta_{ij}\left(1 + \frac{2w}{c^2}\right), \tag{1}$$

where $w$ and $w^i$ are the scalar- and the vector gravitational potentials respectively. Our interest is in the gravitational time delay that can be computed from the null condition, $ds^2 = 0$, along the light-ray. Writing $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and defining the coordinates as $(ct, x, y, z)$, we get

$$dt^2 = \frac{1}{c^2}d\boldsymbol{x}^2 + \left(h_{00} + \frac{2}{c}h_{0i}\frac{dx^i}{dt} + \frac{1}{c^2}h_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}\right)dt^2.$$

Considering $|h_{\mu\nu}| \ll 1$, to first order Taylor expansion, the above equation becomes

$$dt \approx \frac{|d\boldsymbol{x}|}{c} + \frac{|d\boldsymbol{x}|}{2c}\left(h_{\mu\nu}n^\mu n^\nu\right), \tag{2}$$

where we have inserted $dx^i/dt = cn^i$ from the unperturbed light-ray equation, $\boldsymbol{x}(t) = \boldsymbol{x}_0 + \boldsymbol{n}c(t - t_0)$ and $n^\mu \equiv (1, \boldsymbol{n})$. For our metric (1), the Time Transfer Function (TTF), $\mathcal{T}(t_0, \boldsymbol{x}_0; \boldsymbol{x}) \equiv t - t_0$ with $ds = |d\boldsymbol{x}|$ reads

$$\begin{aligned}\mathcal{T}(t_0, \boldsymbol{x}_0; \boldsymbol{x}) &= \frac{R}{c} + \frac{1}{2c}\int_{s_0}^{s}\left(h_{\mu\nu}n^\mu n^\nu\right)ds \\ &= \frac{R}{c} + \frac{2}{c^3}\int_{s_0}^{s}\left(w - \frac{2}{c}\boldsymbol{w}\cdot\boldsymbol{n}\right)ds,\end{aligned} \tag{3}$$

where $R$ is the Euclid distance from $\boldsymbol{x}_0$ (where a light signal is send at time $t_0$) to an observer at $\boldsymbol{x}$ (the reception time is $t$). The TTF allows the computation of $t$ if $t_0, \boldsymbol{x}_0$ and $\boldsymbol{x}$ are given. In one word, TTF is just propagation time of light in gravitational field. Because $t$ is coordinate time, the TTF as well as the time delay should be a coordinate-dependent quantity.

## 3. A single gravitating body at rest

We consider first a single body at rest at the origin of our coordinate system. Space–time outside of the body is assumed to be stationary. Then the metric potentials outside the body take the form [10]

$$w = G\sum_{l\geq 0}\frac{(-1)^l}{l!}M_L\partial_L\left(\frac{1}{r}\right), \tag{4}$$

$$w_i = -G\sum_{l\geq 1}\frac{(-1)^l}{l!}\frac{l}{l+1}\varepsilon_{ijk}S_{kL-1}\partial_{jL-1}\left(\frac{1}{r}\right), \tag{5}$$

where $M_L$ and $S_L$ is the mass and spin multipole moment respectively. $L$ is a Cartesian multi-index, $L \equiv i_1 \dots i_l$, and each individual Cartesian index $i_j$ runs over $1, 2, 3$ or $x, y, z$. Correspondingly, the multi-index $L - 1$ indicates $l - 1$ different Cartesian indices. And $r \equiv (x^2 + y^2 + z^2)^{1/2}$ is the Euclid distance from the center of mass to the field point. We now use the Kopeikin-parametrization of the unperturbed light-ray (see Kopeikin [7])

$$\boldsymbol{x}_s = \boldsymbol{d} + \boldsymbol{n}\cdot s \tag{6}$$

with $\boldsymbol{d}\cdot\boldsymbol{n} = 0$, i.e. $\boldsymbol{d} = \boldsymbol{n}\times(\boldsymbol{x}\times\boldsymbol{n}) = \boldsymbol{n}\times(\boldsymbol{x}_0\times\boldsymbol{n})$ is the vector that points from the origin to the point of closest approach of the unperturbed light-ray. We then have $s = \boldsymbol{n}\cdot\boldsymbol{x}_s$ and $r_s \equiv |\boldsymbol{x}_s| = \sqrt{d^2 + s^2}$. Following [7] we can now split the partial derivative with respect to $x^i$ in the form

$$\partial_i = \partial_i^\perp + \partial_i^\parallel \tag{7}$$

with

$$\partial_i^\perp \equiv \frac{\partial}{\partial d_i}, \qquad \partial_i^\parallel \equiv n^i\frac{\partial}{\partial s}. \tag{8}$$

Then, from Eq. (24) in [7]:

$$\partial_L = \sum_{p=0}^{l}\frac{l!}{p!(l-p)!}n^P\partial_{L-P}^\perp\partial_s^p, \tag{9}$$

where $n^P = n^{i_1}\dots n^{i_p}$ and $\partial_s^p = \partial^p/\partial s^p$. Inserting this into expression (3) and decomposing $\mathcal{T}$ as $\mathcal{T}_\mathrm{M} + \mathcal{T}_\mathrm{S}$ we get

$$\begin{aligned}\mathcal{T}_\mathrm{M} = \frac{2G}{c^3}\sum_{l=0}^{\infty}\sum_{p=0}^{l}\frac{(-1)^l}{l!}\frac{l!}{p!(l-p)!}M_L n^P\partial_{L-P}^\perp\Bigg[\partial_s^p\ln\frac{s+r}{s_0+r_0} \\ -\left(\partial_s^p\ln\frac{s+r}{s_0+r_0}\right)\Bigg|_{s=s_0}\Bigg]\end{aligned} \tag{10}$$

for the time delay induced by the mass multipole moments $M_L$ and

$$\begin{aligned}\mathcal{T}_\mathrm{S} = \frac{4G}{c^4}\sum_{l=1}^{\infty}\sum_{p=0}^{l}\frac{(-1)^l}{l!}\frac{l!}{p!(l-p)!}\frac{l}{l+1}\epsilon_{ijk}n^i S_{kL-1}n^P\partial_{jL-P-1}^\perp \\ \times\left[\partial_s^p\ln\frac{s+r}{s_0+r_0} - \left(\partial_s^p\ln\frac{s+r}{s_0+r_0}\right)\Bigg|_{s=s_0}\right]\end{aligned} \tag{11}$$

for the time delay induced by the spin multipole moments $S_L$, since

$$\int_{s_0}^{s}\frac{ds}{r_s} = \ln\frac{s+r}{s_0+r_0}. \tag{12}$$

These results are in agreement with the ones found by Kopeikin [7].

## 4. The TTF for a body slowly moving with constant velocity

Let us now consider the situation where the gravitating body (called A) moves with a constant slow velocity $\boldsymbol{v}_\mathrm{A}$; we will neglect terms of order $v_\mathrm{A}^2$ in the following. Let us denote a canonical coordinate system moving with body A, $X^\alpha = (cT, X^a)$ (see, e.g., [11]) and the corresponding metric potentials by $W$ and $W^a$. The metric tensor in the comoving system is of the form (1) with potentials $W, W^a$ given by Eqs. (4) and (5), but written in terms of comoving coordinates. E.g., the quantity $r$ in (4) and (5) has to be replaced by $R \equiv |\boldsymbol{X}|$, and the spatial derivatives are now with respect to $X^a$. Under our conditions the transformation from comoving coordinates $X^\alpha$ to $x^\mu$ is a linear Lorentz-transformation of the form ($\boldsymbol{\beta}_\mathrm{A} \equiv \boldsymbol{v}_\mathrm{A}/c$):

$$x^\mu = z_\mathrm{A}^\mu(T) + \Lambda_\alpha^\mu X^\alpha \tag{13}$$

with $z_\mathrm{A}^\mu \equiv (0, \boldsymbol{z}_\mathrm{A}(T))$ and $\Lambda_0^0 = 1$, $\Lambda_a^0 = \beta_\mathrm{A}^a$, $\Lambda_0^i = \beta_\mathrm{A}^i$, $\Lambda_a^i = \delta_{ia}$. where $\boldsymbol{z}_\mathrm{A}$ is the global coordinate position vector of body A. A transformation of the comoving metric to the rest-system then yields (see also [11])

$$\begin{aligned}w &= W + \frac{4}{c}\boldsymbol{\beta}_\mathrm{A}\cdot\mathbf{W} \\ w_i &= W v_\mathrm{A}^i + W_i.\end{aligned} \tag{14}$$

One can show (e.g., Zschocke and Soffel [12]) that $R = r_\mathrm{A}(t) + \mathcal{O}(v_\mathrm{A}^2)$. Furthermore,

$$\partial_a = \frac{\partial}{\partial X^a} = \Lambda_a^\mu\frac{\partial}{\partial x^\mu} = \delta_{ai}\partial_i + \mathcal{O}\left(v_\mathrm{A}^2\right), \tag{15}$$