Physics Letters A 379 (2015) 279–283

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Minimal realizations of supersymmetry for matrix Hamiltonians

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ARTICLE INFO

Article history:
Received 5 October 2014
Received in revised form 24 November 2014
Accepted 25 November 2014
Available online 28 November 2014
Communicated by P.R. Holland

Keywords:
Supersymmetry
Intertwining operator
Matrix non-Hermitian Hamiltonian

ABSTRACT

The notions of weak and strong minimizability of a matrix intertwining operator are introduced. Criterion of strong minimizability of a matrix intertwining operator is revealed. Criterion and sufficient condition of existence of a constant symmetry matrix for a matrix Hamiltonian are presented. A method of constructing of a matrix Hamiltonian with a given constant symmetry matrix in terms of a set of arbitrary scalar functions and eigen- and associated vectors of this matrix is offered. Examples of constructing of 2×2 matrix Hamiltonians with given symmetry matrices for the cases of different structure of Jordan form of these matrices are elucidated.

1. Introduction

The matrix models with supersymmetry appear in Quantum Mechanics in several areas: in particular, for spectral design of potentials describing multichannel scattering and the motion of spin particles in external fields. The different cases of such models were proposed in [1–16] and their investigation was undertaken in [17–33] (see also the recent reviews [34,35] and the paper [25] for more details). The systematic study of intertwining relations for $n \times n$ matrix non-Hermitian, in general, one-dimensional Hamiltonians has been performed in [25,33] with intertwining realized by $n \times n$ matrix linear differential operators with nondegenerate coefficients at d/dx in the highest degree. It is shown in [25] that for any matrix intertwining operator of minimal order there is a matrix operator, in general, of different order that intertwines the same Hamiltonians in the opposite direction and such that the products of these operators in different order are the same polynomials of the corresponding Hamiltonians. Also, the related polynomial algebra of supersymmetry is constructed and the problems of minimization (see more details on this problem below) and of reducibility of a matrix intertwining operator are posed in [25]. Some methods of constructing of $n \times n$ matrix intertwining operator of the first order in derivative and of general form were proposed and their interrelations were examined in [33].

In the one-dimensional QM with scalar Hamiltonians the isospectral transformations generally lead to a Nonlinear (Polynomial [36,37]) SUSY algebra. In respect to the SUSY partners there might be an infinite number of intertwining operators which provide the same pair of potentials but different SUSY algebras [38,39]. These intertwining operators differ in factors which are functions of the Hamiltonians themselves. Thus for intertwining operators and the SUSY algebra itself the problem arises to minimize the order of their differential representation — minimizability problem. For some classes of potentials minimized algebras may exist which contain the same partner Hamiltonians but different sets of intertwining operators. In this case the hidden symmetry operators appear [38] being built of products of intertwining operators from different algebras. In the case of scalar Hamiltonians the number of independent minimized algebras cannot exceed two. The similar program has not been yet realized for matrix SUSY partner Hamiltonians although few important steps in this direction were done in [25].

In the present paper the following new results are elaborated:

- The notions of weak and strong minimizability of a matrix intertwining operator are introduced (the notion of weak minimizability is identical to the notion of minimizability from [25] and the notion of strong minimizability is a new notion).
- Criterion of strong minimizability of a matrix intertwining operator is revealed (criterion of weak minimizability was found in [25]).
- Criterion and sufficient condition of existence of a constant symmetry matrix for a matrix Hamiltonian are presented.
- A method of constructing of a matrix Hamiltonian with a given constant symmetry matrix in terms of a set of arbitrary scalar functions and eigen- and associated vectors of this matrix is offered.

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http://dx.doi.org/10.1016/j.physleta.2014.11.052
0375-9601/

- Examples of constructing of 2×2 matrix Hamiltonians with given symmetry matrices for the cases of different structure of Jordan form of these matrices are elucidated.

The basic notations for intertwining operator algebra are defined in Section 2. In Section 3 we explain the motivation for introduction of the notions of weak and strong minimizabilities and give their definitions. The previous results on minimizability are briefly formulated: the criterion of complete weak minimizability is given for an $n \times n$ matrix intertwining operator from [25,40] and it is noted that partial case $n = 1$ of this criterion coincides with the criterion of complete minimizability for a scalar intertwining operator from [38,39]. At the end of this section the criterion of partial strong minimizability from the right for a matrix intertwining operator is presented. In Section 4 the sufficient condition of existence of a constant symmetry matrix for a matrix Hamiltonian is found. This condition provides us with the opportunity to receive the useful in practice formula for constructing of a matrix Hamiltonian with a given constant symmetry matrix. The two examples of constructing of arbitrary 2×2 matrix Hamiltonian with a given symmetry matrix A are presented. In the first of these examples there are two different eigenvalues for the matrix A and in the second of these examples normal (Jordan) form of A is a Jordan block. In Conclusion we outline the perspectives for further studies of the criteria of minimizabilities and hidden symmetries induced by extended SUSY algebras.

2. Basic definitions and notation

Let us give briefly the definitions and notations needed for formulation of key results of the Letter. We deal with two $n \times n$ matrix Hamiltonians defined on the entire axis,

$$H_+ = -I_n\partial^2 + V_+(x), \quad H_- = -I_n\partial^2 + V_-(x), \quad \partial \equiv \frac{d}{dx},$$

with the identity matrix I_n and two square matrices $V_+(x)$ and $V_-(x)$, whose elements are smooth and, in general, complex-valued functions. These Hamiltonians are supposed to be *intertwined* by a matrix-valued linear differential operator Q_N^-, so that

$$Q_N^- H_+ = H_- Q_N^-, \qquad Q_N^- = \sum_{j=0}^{N} X_j^-(x)\partial^j, \tag{1}$$

where coefficients $X_j^-(x)$, $j = 0, \dots, N$ are square matrices of n-th order, whose elements are smooth and, in general, complex-valued functions.

It is not hard to check [25] that intertwining (1) leads to the following consequences:

$$X_N^-(x) = \mathrm{Const},$$
$$V_-(x) = X_N^- V_+(x)\left(X_N^-\right)^{-1} + 2X_{N-1}^{-\prime}(x)\left(X_N^-\right)^{-1}, \tag{2}$$

provided that $\det X_N^- \neq 0$ which is assumed for all further discussion.

Let us elucidate the structure of intertwining operator kernel and define the transformation vector-functions. As follows from (1) the kernel of Q_N^- is an invariant subspace for H_+:

$$H_+ \ker Q_N^- \subset \ker Q_N^-.$$

Hence, for any basis $\Phi_1^-(x), \dots, \Phi_d^-(x)$ in $\ker Q_N^-$, $d = \dim \ker Q_N^- = nN$ there is a constant square matrix $T^+ \equiv \|T_{ij}^+\|$ of d-th order such that

$$H_+\Phi_i^- = \sum_{j=1}^{d} T_{ij}^+\Phi_j^-, \quad i = 1, \dots, d.$$

The *canonical basis* in the kernel of an intertwining operator Q_N^- is defined by the matrix T^+ in a normal (Jordan) form. *Transformation vector-functions* represent elements of a canonical basis.

If a Jordan form of the matrix T^+ has block(s) of order higher than one, then the corresponding canonical basis contains not only formal vector-eigenfunctions of H_+ but also formal associated vector-function(s) of H_+ [41]. A finite or infinite set of vector-functions $\Phi_{m,i}(x)$, $i = 0, 1, 2, \dots$ is called a *chain of formal associated vector-functions* of H_+ for a spectral value λ_m if

$$H_+\Phi_{m,0} = \lambda_m\Phi_{m,0}, \quad \Phi_{m,0}(x) \not\equiv 0,$$
$$(H_+ - \lambda_m I_n)\Phi_{m,i} = \Phi_{m,i-1}, \quad i = 1, 2, 3, \dots.$$

3. Weak and strong minimizability of a matrix intertwining operator

In what follows we introduce the notions of weak and strong minimizabilities and give their definitions.

Let us multiply Q_N^- by a polynomial of the Hamiltonian,

$$Q_N^-\left[\sum_{l=0}^{L} A_l H_+^l\right] \qquad \left(\left[\sum_{l=0}^{L} A_l H_-^l\right] Q_N^-\right),$$

where A_l, $l = 0, \dots, L$ are either complex numbers or constant symmetry matrices for H_+ (H_-). Then it is evident that such operators intertwine the same Hamiltonians:

$$\left\{Q_N^-\left[\sum_{l=0}^{L} A_l H_+^l\right]\right\} H_+ = Q_N^- H_+ \left[\sum_{l=0}^{L} A_l H_+^l\right] = H_-\left\{Q_N^-\left[\sum_{l=0}^{L} A_l H_+^l\right]\right\}$$

$$\left(\left\{\left[\sum_{l=0}^{L} A_l H_-^l\right] Q_N^-\right\} H_+ = \left[\sum_{l=0}^{L} A_l H_-^l\right] H_- Q_N^- = H_-\left\{\left[\sum_{l=0}^{L} A_l H_-^l\right] Q_N^-\right\}\right).$$

Thus, these polynomials of Hamiltonians are redundant in the program of spectral design and must be removed to minimize the order of intertwining operators.

Let us now present the formal definitions of weak and strong minimizability of a matrix intertwining operator.

Definition 1. An intertwining operator Q_N^- is called *weakly minimizable* if it can be factorized in the form

$$Q_N^- = P_M^-\left[\sum_{l=0}^{L} a_l H_+^l\right] \equiv \left[\sum_{l=0}^{L} a_l H_-^l\right] P_M^-,$$
$$a_l \in \mathbb{C},\ l = 0, \dots, L,\ a_L \neq 0,\ 1 \leqslant L \leqslant \frac{N}{2}, \tag{3}$$

where P_M^- is an $n \times n$ matrix linear differential operator of M-th order, $M = N - 2L$ that intertwines the Hamiltonians H_+ and H_-, so that $P_M^- H_+ = H_- P_M^-$. Otherwise, the operator Q_N^- is called *weakly non-minimizable*.

Definition 2. An intertwining operator Q_N^- is called *strongly minimizable from the right* (*from the left*) if it can be factorized in the form

$$Q_N^- = P_M^-\left[\sum_{l=0}^{L} A_l H_+^l\right] \qquad \left(Q_N^- = \left[\sum_{l=0}^{L} A_l H_-^l\right] P_M^-\right), \tag{4}$$