Physics Letters A 379 (2015) 299–307

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Modulational and oscillatory instabilities of Bose–Einstein condensates with two- and three-body interactions trapped in an optical lattice potential

S. Sabari [a], K. Porsezian [a,*], R. Murali [b]

[a] Department of Physics, Pondicherry University, Puducherry-605014, India
[b] Photonics, Nuclear and Medical Physics Division, School of Advanced Sciences, VIT University, Vellore-632 014, Tamilnadu, India

ARTICLE INFO

Article history:
Received 4 October 2013
Received in revised form 3 December 2013
Accepted 17 December 2013
Available online 18 November 2014
Communicated by A.R. Bishop

Keywords:
Variational approach
Gross–Pitaevskii equation
Three-body interaction
Optical potential
Modulational instability
Oscillatory instability

ABSTRACT

We explain how the modulational and oscillatory instabilities can be generated in Bose–Einstein condensates (BECs) with two- and three-body interactions trapped in a periodic optical lattice with driving harmonic potential. We solve a cubic–quintic Gross–Pitaevskii (GP) equation with external trapping potentials by using both analytical and numerical methods. Using the time-dependent variational approach, we derive and analyze the variational equations for the time evolution of the amplitude and phase of modulational perturbation, and effective potential of the system. Through the effective potential, we obtain the modulational instability condition of the BECs with two- and three-body interactions and shown the effects of the optical potential on the dynamics of the system. We perform direct numerical simulations to support our analytical results, and good agreement is observed.

1. Introduction

In recent years, since the first successful experimental realization of Bose–Einstein condensates (BECs) [1], much experimental [2–4] and theoretical [5] interest in this field has been attracted. In particular, the investigations of the dynamics and properties of BECs in optical lattice (OL) potentials have been a significant interest in this topic. The OL is created by two counterpropagating laser beams forming a standing wave interference pattern. In these experiments, the properties of the atoms are characterized by the depth and the period of this optically induced potential. The density of atoms in an OL can be tuned from weak to strong by using the optical Feshbach resonance technique [2]. Consequently, BECs in OL potentials have been found useful in investigating many physical phenomena such as Josephson effect [6], Bloch oscillations [2,7], Landau–Zener tunneling [8], solitons [9], quantum phase transitions of the Mott insulator type [10], superfluid and dissipative dynamics [11], phase diagram [12], and nonlinear dynamics of a dipolar [13] or spinor BEC [14].

At ultracold temperatures, the dynamics and properties of BECs can be well described by the mean-field Gross–Pitaevskii (GP) equation. Nonlinear terms arise in the GP equation to account for the effect of interatomic interactions in the condensates. Two-body interatomic interactions in BECs are modeled through the s-wave scattering length, a_s, which may be either negative or positive, meaning that the interaction is attractive or repulsive, respectively [15]. At low enough temperatures the magnitude of the scattering length a_s is much lower than the thermal de Broglie wavelength, and the exact shape of the two-atom interaction is unimportant. On the other hand, if the atom density is considerably higher the three-body interaction can start to play an important role [16–18]. The strength and sign of the atomic scattering length can be varied by tuning the external magnetic field near Feshbach resonance [19]. This indicates that the interaction strength can be controlled by using different experimental devices. When the sign of the s-wave scattering length a_s is positive, the interaction in the BEC is repulsive (defocussing nonlinearity). As is well known, in the presence of OL, the repulsive BECs can give rise to stable localized matter-wave states in the form of gap solitons. These BEC gap solitons were predicted theoretically [16,20–22] and demonstrated experimentally [4,23]. Gap solitons are represented by stationary solutions to the respective GP equation, with the

* Corresponding author. Tel.: +914132654403; fax: +914132655183.
E-mail address: ponzsol@yahoo.com (K. Porsezian).

http://dx.doi.org/10.1016/j.physleta.2013.12.047
0375-9601/

eigenvalue (chemical potential) located in a finite bandgap of the OL-induced spectrum [23]. In the BECs with attractive interactions (focusing nonlinearity), solitons have been realized in the ground state of the condensate. Such solitons were created in condensates of ^{7}Li [24] and ^{85}Rb [25] atoms, with the sign of the atomic interactions switched to attraction by means of the Feshbach-resonance technique (in the latter case, the solitons were observed in a post-collapse state of the condensate). In the presence of a periodic potential, such solitons should exist too, with the chemical potential falling in the semi-infinite gap of the spectrum, as first shown in the context of the optical setting [26], and later demonstrated in detail in the framework of GP equation [27,28].

The phenomenon of modulational instability (MI) was first predicted for waves in deep water and for electromagnetic waves in nonlinear media [29]. It has been observed in many branches of physics such as nonlinear optics [30], plasma physics [31], magnetics [32], fibers [33], long Josephson junction [34] and BEC [35], etc. It indicates that, due to the interplay between nonlinearity and the dispersive effects, a small perturbation on the envelope of a plane wave may induce an exponential growth of the wave amplitude, resulting in a break-up of the carrier wave into a train of localized waves [36]. BECs in the presence of an OL (more specifically, a sinusoidal external potential) have shown some intriguing demonstrations such as the dephasing and localization that occur because of the effect of MI. It has been studied both experimentally [37–39] and theoretically [40–45]. Recently, many investigations have been devoted to the MI of both single-component BECs and double-species BECs in OLs [46,47]. Moreover, numerous studies with relation to the MI have attracted much interest, as the MI is an indispensable mechanism for understanding the relevant dynamic processes in BEC systems, which include domain formation [48], generation and propagation of solitonic waves [49], quantum phase transition [50], etc. In the specific case of condensates trapped in a periodic potential and driven by a harmonic magnetic field, the MI has been observed experimentally in Ref. [39], following the theoretical analysis presented in Ref. [40]. However, only few efforts have been devoted to the investigation of MI domains and time-dependence of MI of BECs with optical and harmonic potentials.

In this paper, we extend our earlier work [51] by including three-body interaction and revisit MI for the GP equation with OL potential. We intend to study the effect of the OL potential on the dynamics of the BECs with two- and three-body interactions. We aim to find the explicit time dependent criteria for MI as well as MI domains of such a system by using the time-dependent variational approach (TDVA) and numerical methods [17]. For this aim, we perform a modified lens-type transformation which converts our initial GP equation, with space-dependent sinusoidal and parabolic potentials, into a GP equation with only the OL potential and with a time-dependent coefficients of cubic nonlinearity and constant coefficient of quintic nonlinearity. The resultant GP equation is then treated by the TDVA to propose not only the MI conditions but also the time-dependence of the perturbation parameters. The work is structured as follows. In Section 2, we present the theoretical model that describes the condensates understudy. Section 3 is devoted to the mathematical framework in which we derive the MI conditions of the system through the variational approach. Then, in Section 4, we perform direct numerical integrations to check the validity of the MI conditions found by analytical methods. Finally, we give concluding remarks in Section 5.

2. Theoretical model

At ultra low temperatures, BEC in OL with two- and three-body interaction can be described by the following nonlinear mean field GP equation [47]

$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t) + \frac{m}{2}\left(\omega_\rho^2\rho^2+\omega_x^2x^2\right)\psi(\mathbf{r},t) + \left(V_{\text{opt}}(x)+g_2\left|\psi(\mathbf{r},t)\right|^2+g_3\left|\psi(\mathbf{r},t)\right|^4\right)\psi(\mathbf{r},t), \quad (1)$$

where $\hbar$ is the reduced Planck's constant, m is the mass of the boson, ω_ρ and ω_x, respectively, are the radial and longitudinal frequencies of the anisotropic trap ($\omega_\rho \gg \omega_x$), and ρ denotes the radial distance. The parameters g_2 and g_3 are the strength of the two- and three-body interatomic interactions, respectively. The role of g_2 depends on the s-wave scattering length a_s by $g_2 = 4\pi\hbar^2 a_s/m$. The OL potential is applied only in the longitudinal direction, i.e., $V_{\text{opt}}(x) = V_{\max}\cos^2(kx+\theta)$, here $V_{\max}$ is the effective depth of the optical potential. The parameter θ is an arbitrary phase and $k = 2\pi/\lambda$ is the wavenumber of the OL that can be controlled by varying the angle ϑ between the two counterpropagating laser beams whose interference creates the OL [23,52]. The wavelength of the interference pattern is expressed in terms of the angle between the laser beams by the relation $\lambda = (\lambda_{\text{laser}}/2)\sin(\vartheta/2)$, where λ_{laser} is the wavelength of the laser beams producing the OL. We take $\theta = 0$ and $\psi(\mathbf{r},t) = \phi_0(\rho)\phi(x,t)$ where $\phi_0 = \sqrt{\frac{1}{\pi a_\perp^2}}\exp(-\frac{\rho^2}{2a_\perp^2})$, with $\rho = \sqrt{x^2+y^2}$ and $a_\perp = \sqrt{\hbar/m\omega_\perp}$, is the ground state of the radial equation

$$-\frac{\hbar^2}{2m}\nabla_\rho^2\phi_0 + \frac{m}{2}\omega_\perp^2\rho^2\phi_0 = \hbar\omega_\perp\phi_0. \quad (2)$$

Multiplying both sides of the GP equation (1) by ϕ_0^* and integrating over the transverse variable ρ, we can get the following quasi-1D GP equation:

$$i\hbar\frac{\partial\phi(x,t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m}{2}\omega_x^2x^2 + V_{\max}\cos^2(kx) + \frac{g_2}{2\pi a_\perp^2}\left|\phi(x,t)\right|^2 + \frac{g_3}{3\pi^2 a_\perp^4}\left|\phi(x,t)\right|^4\right)\phi(x,t). \quad (3)$$

It is more convenient to use the above equation (3) into a dimensionless form [53]. For this purpose we make the transformation of variables as $T = t\nu$, $X = xk$, $\phi \sim \phi\sqrt{2a_{s0}\omega_\perp/\nu}$, where $\nu = E_R/\hbar$, $\alpha = \omega_x^2/4\nu^2$ and $E_R = \hbar^2k^2/2m$, then, we can get the following normalized 1D GP equation with harmonic and optical potentials:

$$i\frac{\partial\phi(X,T)}{\partial T} = -\frac{\partial^2\phi(X,T)}{\partial X^2} + \left(\alpha X^2 + V_s\cos^2(X)\right)\phi(X,T) + g_0\left|\phi(X,T)\right|^2\phi(X,T) + \chi\left|\phi(X,T)\right|^4\phi(X,T), \quad (4)$$

where $V_s = V_{\max}/E_R$, $a_s = g_0 a_{s0}$, with a_{s0} the constant scattering length, $g_0 = \pm 1$ is the sign of the scattering length. In order to obtain the exact analytical condition to investigate the MI for the system of BEC, first we need to find an expression for Eq. (4) with less space-dependent coefficients. We start with a modified lens-type transformation by setting [17,54]

$$\phi(X,T) = \frac{1}{\sqrt{l(T)}}\tilde{\psi}(\tilde{x},\tilde{t})\exp\left(if(T)X^2\right), \quad (5)$$

the time-dependent parameters are chosen as $l(T) = |\cos(2\sqrt{\alpha}T)|$, $\tilde{x} = \frac{X}{l(T)}$, $\tilde{t}(T) = \frac{1}{2\sqrt{\alpha}}\tan(2\sqrt{\alpha}T)$, and $f(T) = -\frac{\sqrt{\alpha}}{2}\tan(2\sqrt{\alpha}T)$. The rescaling signals the existence of negative $\tilde{t}$ and is valid for any $T \neq \frac{(2n+1)\pi}{4\sqrt{\alpha}}$ (where n is a positive integer) in the $(T,\tilde{t})$ plane. We consider the case where T goes from zero to $\pi/(4\sqrt{\alpha})$ to ensure a variation of $\tilde{t}$ from zero to infinity. We will drop the tildes for simplicity. Then Eq. (4), in terms of the new variables x and t, is reduced to