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Effects of phase lag on the information rate of a bistable Duffing oscillator

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ABSTRACT

To utilize noise for systems, which are transmitting or receiving information, the information rate is a necessary metric to consider. The phase lag, which is the difference between the sender (applied forcing) and receiver (the oscillator) phases, has a significant effect on the information rate. However, this phase lag is a nonlinear function of the noise level. Here, the effects of phase lag on the information rate for a Duffing oscillator are examined and comparative discussions are made with phase lag from linear response theory. The phase lag is shown to be an important variable in calculating the information rate.

1. Introduction

Stochastic resonance has been classically identified as a peak in the signal-to-noise ratio [1], when a response measure is plotted against the amplitude of noise. This phenomenon, which has been used to describe the effects of noise on the recurrence of ice ages [2], was first studied in the context of a bistable Duffing oscillator [1] and also shown to have an important effect on the response of a monostable Duffing oscillator [3]. In a recent study, stochastic resonance has been studied in a macroscale, distributed parameter system, a post-buckled beam [4]. The behavior predicted by the integrate-and-fire model for neurons is similar to the behavior shown by a monostable Duffing oscillator [5].

Since Shannon's seminal information theory work [6], information-theoretic techniques have been applied to many different systems. A binary channel has been studied by using information capacity [7]. In experimental work, mutual information has been used to show broadband stochastic resonance in a neuron [8]. The channel capacity has also been used to detect the occurrence of stochastic resonance in a neuron model [9], where it was noted that the location of the channel capacity maximum occurs at a higher noise amplitude than does the maximum of the signal-to-noise ratio (SNR). In the present work, a similar result is obtained. Furthermore, for different phase lags, the peak is found to occur at different noise amplitudes. For a neuron model, aperiodic stochastic resonance has been measured by using mutual information [10]. Also, this measure was used to study a neuron experiencing adaptive stochastic resonance [11]. Simulations have also been carried out with a bistable dynamic system by using an Euler discretization scheme, revealing a single peak in the channel capacity [12]. After converting input and output signals to binary sequences, experimental data from a Schmitt trigger have been shown to exhibit extrema when studied with conditional and Kullback entropies [13]. For weak forcing and noise variance, the behavior of the normal form equation was examined by using mutual information [14]. The Fisher information measure has also been used to study responses of a parallel array of sensors [15]. Entropy measures for several dynamical systems, including a linear oscillator, are discussed in reference [16].

The rest of this article is organized as follows. In the next section, the equations governing the nondimensionalized bistable Duffing oscillator are discussed. Euler–Maruyama simulations are presented in the following section, as well as the approach used to convert the continuous system response into a binary signal.

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x	Oscillator displacement	$\hat{K}$	Nondimensionalized stiffness
x_1	Oscillator position in state space	Ω	Forcing frequency
x_2	Oscillator velocity in state space	$\hat{\Omega}$	Nondimensionalized forcing frequency
c	Viscous damping	ω_n	Natural frequency
ζ	Damping ratio	$W'(\tau)$	White Gaussian noise (derivative of Wiener process)
k_1	Linear stiffness	$\hat{\sigma}$	Nondimensionalized noise amplitude
k_3	Nonlinear stiffness		

The Fokker–Planck equation and cumulant neglect method are presented in the subsequent section. By using the Euler–Maruyama method and the moment evolution equations, the information rate is used to assess the influence of noise on the system response. By considering the phase lag as a parameter in calculating the information rate, a relationship amongst the phase lag, noise amplitude, and information rate is shown. Concluding remarks are collected together in the final section.

2. System equations

The equation of motion governing a bistable Duffing oscillator with mass m, viscous damping c, linear stiffness k_1, nonlinear stiffness k_3, forcing amplitude F, and forcing frequency Ω can be written as

$$m\ddot{x} + c\dot{x} - k_1 x + k_3 x^3 = F\sin(\Omega t) \tag{1}$$

where all of the parameters assume positive values and an overdot indicates differentiation with respect to time t. After dividing Eq. (1) by m and introducing a nondimensional time $\tau = \omega_n t$, the resulting equation takes the form

$$\omega_n^2 \frac{d^2x}{d\tau^2} + 2\zeta\omega_n^2 \frac{dx}{d\tau} - \omega_n^2 x + \frac{k_3}{m}x^3 = \frac{F}{m}\sin\left(\frac{\Omega}{\omega_n}\tau\right) \tag{2}$$

Dividing through by ω_n^2 and introducing primes to indicate differentiation with respect to the nondimensional time τ, the result is

$$x'' + 2\zeta x' - x + \frac{k_3}{k_1}x^3 = \frac{F}{k_1}\sin\left(\frac{\Omega}{\omega_n}\tau\right) \tag{3}$$

Finally, after substituting the nondimensional parameters $\hat{K} = \frac{k_3}{k_1}$, $\hat{F} = \frac{F}{k_1}$, and $\hat{\Omega} = \frac{\Omega}{\omega_n}$, the equation becomes

$$x'' + 2\zeta x' - x + \hat{K}x^3 = \hat{F}\sin(\hat{\Omega}\tau) \tag{4}$$

After including nondimensional noise, the stochastic differential equation (SDE) for the nondimensionalized bistable Duffing equation is modified to

$$x'' + 2\zeta x' - x + \hat{K}x^3 = \hat{F}\sin(\hat{\Omega}\tau) + \hat{\sigma}W'(\tau) \tag{5}$$

2.1. Numerical results

In Eq. (5), the oscillator is subjected to a deterministic forcing $\hat{F}\sin(\hat{\Omega}\tau)$ and a stochastic input $\hat{\sigma}W'(\tau)$. Writing in state-space form, one obtains

$$\begin{cases} \dfrac{dx_1}{d\tau} = x_2 \\ \dfrac{dx_2}{d\tau} = -2\zeta x_2 + x_1 - \hat{K}x_1^3 + \hat{F}\sin(\hat{\Omega}\tau) + \hat{\sigma}W'(\tau), \end{cases} \tag{6}$$

where x_1 and x_2 correspond to the position and velocity, respectively. The white noise term, $W'(\tau)$, is defined as the derivative of Brownian motion. Since Brownian motion (or in the physics literature, the Wiener process) has independent increments, its derivative does not exist with probability one [17]. For this reason, $W'(\tau)$ is a "mnemonic" derivative. Hence, writing Eq. (6) in differential form, one has

$$\begin{cases} dx_1 = x_2 d\tau \\ dx_2 = \left[-2\zeta x_2 + x_1 - \hat{K}x_1^3 + \hat{F}\sin(\hat{\Omega}\tau)\right]d\tau + \hat{\sigma}dW \end{cases} \tag{7}$$

In this form, one no longer has the derivative of Brownian motion but a differential white noise which does exist. This system is integrated as an Itō integral, and the Euler–Maruyama method can be used to obtain numerical solutions for Eq. (7) [18]. These simulations were performed on a desktop computer, with a 4.00 GHz processor, by using MATLAB. Although this code was not optimized for speed, it took approximately 7.6 hours of wall time to run the 200 Euler–Maruyama simulations for each of the 100 noise amplitudes considered and to do the subsequent averaging.

The information rate R is defined as

$$R = H(x) - H_y(x), \tag{8}$$

where the Shannon entropy is $H(x) = -\sum_i p_i \log_2 p_i$ and the conditional entropy is $H_y(x) = -\sum_{i,j} p(i,j)\log_2 p_i(j)$, where $p_i(j) = \frac{p(i,j)}{\sum_j p(i,j)}$. Since the Duffing oscillator response is a continuous time response, one needs to convert the oscillator's position response into a binary output, before the associated information rate can be computed. A natural choice is to represent displacements below zero with "0" and displacements above zero with "1". With this choice, the positive piece of an input sine wave from 0 to π would be considered as one instance of "1". Thus, averaging over this range is sufficient to convert the continuous signal into a binary signal. However, given the inherent delay in an input-output relationship, care must be exercised in choosing a phase lag for the continuous output signal, before converting it into a binary one. In this paper, phase lag constants for different noise amplitudes are considered and their effects are examined. An example of this process for the no noise and no phase lag case is presented in Fig. 1. In the first part of this figure, the forcing input, in black color, and the output response, in grey color, are both plotted. In the middle part of this figure, the binary conversion scheme is implemented as previously discussed. In the bottom part of this figure, the binary displacements have been averaged over sections of one half period of the forcing frequency. After this was done, if the averaged binary displacement over that length of time was above "0", it was converted to a bit value of "1". If the averaged binary displacement over that length of time was below "0", it was converted to a bit value of "0". By shifting the half period sections to be averaged (i.e., by changing the phase lag), different averages and subsequent bit values can be obtained and studied.

In Figs. 2 and 3, the computed mean information rate for 200 simulations is shown, over a noise level range. The effects of different phase lag values can be seen in both figures. In Fig. 4, the maximal information rate is plotted against the noise amplitude, along with the phase lag amount that maximizes the information rate. Interestingly, the phase lag graph is not monotonic. When a case with no phase lag is considered, a double peak can be observed in the information rate as shown in Fig. 5. The position of