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On the Lagrangian theory for rotating charge in the Maxwell field

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ABSTRACT

We justify the Hamilton least action principle for the Maxwell–Lorentz equations coupled with the equations of motion of Abraham's rotating extended electron. The main novelty in the proof is the application of the variational Poincaré equations on the Lie group $SO(3)$.

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1. Introduction

We justify Hamilton's least action principle for the system of Maxwell–Lorentz equations coupled with the equations of motion of a rotating charged particle. Our main contribution is the variational derivation of the *Lorentz torque equation*, see Eq. (1.3) below.

First recall the case of a finite system of material points (q_i, m_i). The *angular momentum* is defined by

$$M := \sum q_i \wedge p_i := \sum q_i \wedge m_i \dot{q}_i. \tag{1.1}$$

By the second and the third Newton's laws this implies

$$\dot{M} = \sum q_i \wedge \dot{p}_i = \sum q_i \wedge F_i = \sum q_i \wedge F_i^{\text{ext}} = T, \tag{1.2}$$

where T is called the *external force torque*. Our aim is to derive the similar Abraham *non-relativistic torque equation* for a charged rigid body in the Maxwell field:

$$I\dot{\omega} = e \int (x-q) \wedge \left[E + E^{\text{ext}} + \left(\dot{q} + \omega \wedge (x-q)\right) \wedge \left(B + B^{\text{ext}}\right)\right] \times \rho(x-q)dx. \tag{1.3}$$

Here I is the moment of inertia, ω is the vector angular velocity, $\rho(x)$ is a distribution of the charge, and the right hand side is the torque of the Lorentz force. Note that in our case the fields E and B are generated by the motion of the charged body, and E^{ext}, B^{ext} are external fields.

Formally, the rigid body can be considered as an infinite system of material points. However Eq. (1.3) cannot be obtained directly from (1.2) since we cannot correctly take into account all the forces of mutual interaction between the different pieces of the rigid body. That is why we look for a different approach to the derivation of (1.3). We show that (1.3) follows from the Hamilton variational least action principle with the standard interaction term $-A_0\rho + \vec{A} \cdot \vec{j}$ in the Lagrangian density, $(A_0, \vec{A})$ being the potential of the fields, see (3.1), (3.2) below.

Let us comment on previous works. For the free rigid body (when $E, E^{\text{ext}}, B, B^{\text{ext}} = 0$) Eq. (1.3) reduces to the Euler's equations which have been obtained from the variational principle first by Poincaré [12]. In [1], this result has been extended to an external force field with an axial symmetry.

Eq. (1.3) is well recognized since Abraham's works [2,3]. In [2, Section 11] Abraham computed the Lagrangian of the Maxwell–Lorentz equations as integral of $-A_0\rho + \vec{A} \cdot \vec{j}$ for standing rotating spherically symmetric electron subject to external fields obeying very special symmetry conditions. In this case the Lagrangian depends only on one variable ω, the angular velocity. However, derivation of the torque equation (1.3) from the variational *Hamilton least action principle* remained an open question.

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The main goal of the Nodvik's paper [11] is a variational derivation of the Lorentz-covariant dynamics for the relativistic rotating charged particle in the Maxwell field, and the proof of the corresponding conservation laws. The system of Nodvick's equations is overdetermined, since they do not include rotational bare inertia. The situation was improved by Appel and Kiessling in [4], where they develop the theory for the relativistic rotating particle introducing a renormalization limit.

We propose an invariant derivation of the non-relativistic Abraham equation (1.3) from the Hamilton least action principle relying on the Poincaré equations [1,12] on the Lie group $SO(3)$.

The new interest for the rather old Abraham model is caused by the fact that a broad class of models of this type display *soliton-type asymptotics* and *scattering behavior* as it was discovered in recent years, see e.g. [5–7]. Lagrangian and Hamiltonian structure of the models play a significant role in these methods, so it is of a considerable interest and importance, to include the Abraham model with rotating charge into the class of Lagrangian systems.

The plan of our article is as follows. In Section 2 we state the Maxwell–Lorentz equations for the rotating charge, and in Section 3 we introduce the corresponding Lagrangian functional and formulate the Hamilton principle. In Section 4 we formulate our main result. In Section 5 we deduce the Lorentz force and the Maxwell equations. In Section 6 we deduce the Lorentz torque equation, and in Appendices A and B we collect some auxiliary calculations.

2. Maxwell–Lorentz equations

The Maxwell field consists of the electric field $E(x,t)$ and the magnetic field $B(x,t)$ generated by a motion of a rotating charge. The external fields E^{ext} and B^{ext} are generated by the corresponding external charges and currents. Let the rotating charge be centered at the position q with the velocity $\dot{q}$. For simplicity we assume that the mass distribution, $m\rho(x)$, and the charge distribution, $e\rho(x)$, are proportional to each other. Here m is the total mass, e is the total charge, and we use a system of units such that $m=1$, $e=1$.

For the coupling function ρ we apply everywhere below the conditions of regularity and spherical symmetry

$$\rho(x)=\rho_r\big(|x|\big),\quad \rho\in C_0^\infty\big(\mathbb{R}^3\big). \tag{2.1}$$

Then the system of Maxwell–Lorentz equations with the rotating charge reads, see [13]

$$\dot{E}=\nabla\wedge B-\big(\dot{q}+\omega\wedge(x-q)\big)\rho(x-q), \tag{2.2a}$$

$$\dot{B}=-\nabla\wedge E, \tag{2.2b}$$

$$\nabla\cdot E(x,t)=\rho\big(x-q(t)\big), \tag{2.3a}$$

$$\nabla\cdot B(x,t)=0, \tag{2.3b}$$

$$\ddot{q}=\int\big[E+E^{\text{ext}}+\big(\dot{q}+\omega\wedge(x-q)\big)\wedge\big(B+B^{\text{ext}}\big)\big]\times\rho(x-q)dx, \tag{2.4}$$

$$I\dot{\omega}=\int(x-q)\wedge\big[E+E^{\text{ext}}+\big(\dot{q}+\omega\wedge(x-q)\big)\wedge\big(B+B^{\text{ext}}\big)\big]\times\rho(x-q)dx, \tag{2.5}$$

where $\omega(t)$ is the vector angular velocity of the particle rotation (see Appendix A), and I is the moment of inertia defined by

$$I=\frac{2}{3}\int x^2\rho(x)dx. \tag{2.6}$$

Eqs. (2.2) are Maxwell equations with the corresponding charge density and current, and Eqs. (2.3) are the constraints. The back reaction of the field onto the particle is given through the Lorentz force equation (2.4), and the Lorentz torque equation (2.5) deals with rotational degrees of freedom.

Note that in [10], the direct proofs of the corresponding conservation laws are presented.

3. Lagrangian functional and the Hamilton principle

Our main goal is to deduce equations (2.2)–(2.5) from the Hamilton least action principle. First let us introduce *electromagnetic potentials* $\mathcal{A}=(A_0,A)$, $\mathcal{A}^{\text{ext}}=(A_0^{\text{ext}},A^{\text{ext}})$:

$$B=\nabla\wedge A,\qquad E=-\nabla A_0-\dot{A}. \tag{3.1}$$

$$B^{\text{ext}}=\nabla\wedge A^{\text{ext}},\qquad E^{\text{ext}}=-\nabla A_0^{\text{ext}}-\dot{A}^{\text{ext}}. \tag{3.2}$$

Next we define the Lagrangian

$$\begin{aligned}L(\mathcal{A},q,R,\dot{\mathcal{A}},\dot{q},\dot{R})&=\frac{1}{2}\int\big(E^2-B^2\big)dx+\frac{1}{2}\dot{q}^2+\frac{1}{2}I\omega^2\\&\quad-\int\big[A_0+A_0^{\text{ext}}\big]\rho(x-q)dx\\&\quad+\int\big(\dot{q}+\omega\wedge(x-q)\big)\cdot\big[A+A^{\text{ext}}\big]\rho(x-q)dx,\end{aligned} \tag{3.3}$$

where E, B are expressed in terms of $\mathcal{A}$, $\dot{\mathcal{A}}$ by (3.1), and $\omega=\mathcal{J}^{-1}\dot{R}R^{-1}$ by (A.2). The last two integrals represent the standard interaction term

$$\int\big[\big(A_0+A_0^{\text{ext}}\big)\rho-j\cdot\big(A+A^{\text{ext}}\big)\big]dx$$

in view of (A.1).

Remark 3.1. This Lagrangian functional does not depend on R due to the spherical symmetry of the charge and mass distributions. Respectively, the dynamical equations (2.2)–(2.5) do not involve R as well. In the case of nonradial densities the Lagrangian and the equations should involve R. Moreover, the moment of inertia (2.6) becomes a matrix in this case.

The corresponding action functional has the form

$$S=S(\mathcal{A},q,R):=\int_{t_1}^{t_2}L\big(\mathcal{A}(t),q(t),R(t),\dot{\mathcal{A}}(t),\dot{q}(t),\dot{R}(t)\big)dt. \tag{3.4}$$

Then the Hamilton least action principle reads

$$\delta S(\mathcal{A},q,R)=0, \tag{3.5}$$

where the variation is taken over $\mathcal{A}(t)$, $q(t)$, $R(t)$ with the boundary conditions

$$(\delta\mathcal{A},\delta q,\delta R)|_{t=t_1}=(\delta\mathcal{A},\delta q,\delta R)|_{t=t_2}=0. \tag{3.6}$$

4. Regular solutions and main result

Below we state our results for *regular solutions* of the system (2.2)–(2.5). This means that $q\in C^2(\mathbb{R},\mathbb{R}^3)$, $\omega\in C^1(\mathbb{R},\mathbb{R}^3)$, and all the involved functions and fields/potentials are sufficiently smooth and have (with all the necessary derivatives) a sufficient decay as $|x|\to\infty$ so that the partial integrations below are allowed.

Our main result is the following theorem.

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