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Tomographic causal analysis of two-qubit states and tomographic discord

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ABSTRACT

We study a behavior of two-qubit states subject to tomographic measurement. In this Letter we propose a novel approach to definition of asymmetry in quantum bipartite state based on its tomographic Shannon entropies. We consider two types of measurement bases: the first is one that diagonalizes density matrices of subsystems and is used in a definition of tomographic discord, and the second is one that maximizes Shannon mutual information and relates to symmetrical form quantum discord. We show how these approaches relate to each other and then implement them to the different classes of two-qubit states. Consequently, new subclasses of X-states are revealed.

1. Introduction

In general, measurements irreversibly change a state of quantum system. Quantum tomography is an experimental method, which restores a complete information about an unknown quantum state using preparation of set of its copies and measurements statistics obtained in different bases.

The main feature of quantum tomography is a complete characterization of quantum states and processes *directly* from experimental data. Quantum states of light were completely characterized via the method of balanced homodyne detection (BHD) [1]. These works inspired series of new experiments [2] as well as intensive theoretical work on analysis and improvement of the BHD setup [2,3]. Moreover, quantum tomography was used for characterization of quantum states of current (voltage) in the Josephson junction [4].

On the other hand, quantum tomography is an original picture of quantum mechanics, where quantum states are described in terms of nonnegative probability distributions functions [5–7]. Quantum tomography is equivalent to other approaches to quantum mechanics, and tomograms are directly related to quasi-probability distribution functions [8,9].

One of the areas, where quantum tomography is of interest, is consideration of correlation properties in bipartite states. As a results of purely probabilistic description of states, tomographic version of the Shannon entropy [10] and the Rényi [11,12] entropy naturally appear. Being a bridge between classical information theory and quantum information theory [13], it allows to use some well-known inequalities for Shannon and Rényi entropies for investigation of novel properties of quantum states [10–14]. Recently, tomographic approach to quantum discord was suggested [15]. In particular, tomographic discord for two-qubit X-states was considered. This analysis posed an important problem of relation between original [16] and tomographic discords.

Another interesting question, posed in Ref. [17], is about the role of asymmetry between parties of bipartite state in respect to its properties. Due to such asymmetry, decoherence acting on different parties leads to different rates of correlation decay, so the question about robustness of parties appears. For the purpose of asymmetry investigation the method of quantum causal analysis was proposed [18]. It was successfully implemented to two- [19] and three- [20] qubit states and atom-field interaction [21], where interesting conclusions were made.

In the current Letter we combine quantum causal analysis with quantum tomography. We obtain two novel measures of bipartite state asymmetry, based on tomographic discord and symmetric version [22] of quantum discord. We show that tomographic discord is not greater than symmetric quantum discord. For a demonstration of obtained results we consider the simplest case of bipartite system, and show that even for them nontrivial phenomena occur.

The Letter is organized as follows. We start from brief consideration of quantum causal analysis in Section 2 and quantum tomography in Section 3. In Section 4 we show how quantum causal analysis can be modified via tomography. In Section 5 we implement tomographic causal analysis different classes of two-qubit states. The results of the Letter are summed up in Section 6.

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2. Quantum causal analysis

Quantum causal analysis [18–21] is a formal method for a treatment of informational asymmetry between parties of bipartite states. The term "causal" comes from classical causal analysis (see, e.g. [23]), where a such asymmetry can be related to real causal connection between two processes. In the quantum domain the conception of causality usually is considered in framework of channels [24] or probability wave propagation [25]. Nevertheless, it is convenient to introduce formal definitions of "causes" and "effects" in bipartite states, however, one should understand them only as labels.[1]

The idea of quantum causal analysis is the following. Consider a bipartite quantum system AB in the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. It is described via density operator $\hat{\rho}_{AB} \in \Omega(\mathcal{H}_{AB})$ with $\hat{\rho}_A = \mathrm{Tr}_B\, \hat{\rho}_{AB} \in \Omega(\mathcal{H}_A)$ and $\hat{\rho}_B = \mathrm{Tr}_A\, \hat{\rho}_{AB} \in \Omega(\mathcal{H}_B)$ being corresponding density operators of subsystems. Here $\Omega(\mathcal{H})$ is the set of positive operators of unit trace (density operators) in a Hilbert space $\mathcal{H}$. The basic quantity of quantum information theory is the von Neumann entropy given by

$$S_X \equiv S[\hat{\rho}_X] = -\mathrm{Tr}[\hat{\rho}_X \log \hat{\rho}_X], \quad X \in \{A, B, AB\}.$$

Here we restrict our consideration to finite dimensional Hilbert spaces and take the logarithm to base 2 (*i.e.*, we measure entropy in bits).

The amount of correlations between A and B is given by the (symmetric) quantum mutual information

$$I_{AB} = S_A + S_B - S_{AB}, \quad I_{AB} = I_{BA}. \tag{1}$$

To describe a possible asymmetry of correlations we introduce a pair of independence functions

$$i_{A|B} = \frac{S_{AB} - S_B}{S_A} = 1 - \frac{I_{AB}}{S_A},$$
$$i_{B|A} = \frac{S_{AB} - S_A}{S_B} = 1 - \frac{I_{AB}}{S_B},$$

which have the following properties: (i) they take values from -1 to 1 and the less $i_{Y|X}$ is, the stronger X defines Y ($i_{Y|X} = -1$ corresponds to maximal quantum correlations, $i_{Y|X} = 0$ corresponds to Y being a classical one-valued function of X, and $i_{Y|X} = 1$ corresponds to Y being independent from X); (ii) negative values correspond to negative conditional entropy and imply a presence of entanglement between partitions; (iii) for all pure entangled states $\hat{\rho}_{AB} = |\Psi\rangle_{AB}\langle\Psi|$ the both independence functions take minimal values ($i_{A|B} = i_{B|A} = -1$); (iv) in general, for mixed states relation $i_{A|B} \neq i_{B|A}$ holds.

Further, we can introduce the following formal definitions: in bipartite state $\hat{\rho}_{AB}$ with $S_A \neq S_B$ the party A is the "cause" and B is the "effect" if $i_{A|B} > i_{B|A}$. Vice versa, one has B being the "cause" and A being the "effect" if $i_{B|A} > i_{A|B}$.

Finally, we need to introduce a measure of asymmetry based on independence functions. In the current Letter it is convenient to use difference

$$d_{AB} = i_{A|B} - i_{B|A} = I_{AB}\frac{S_A - S_B}{S_A S_B}, \quad d_{AB} \in (-2, 2). \tag{2}$$

The zero value is obtained for symmetric or non-correlated states, while the extreme values are obtained in cases when the entropy of one subsystem tends to zero, while the entropy of another does not, and mutual information takes the maximal possible value which is doubled entropy of the first subsystem.

[1] The question about connection between asymmetry in bipartite states and real causality is interesting, however, it is beyond the present work scope.

3. Quantum tomography

Quantum tomography suggests physical picture of quantum mechanics as well as it has interesting mathematical structure. Mathematical aspects of quantum tomography are well understood in terms of group theory [26], C^* algebra [27] and groupoids [28].

Following [26], we define quantum tomograms thought mapping of $\hat{\rho} \in \Omega(\mathcal{H})$ on a parametric set of probability distribution functions

$$\hat{\rho} \in \Omega(\mathcal{H}) \xrightarrow{\mathrm{G}(g)} \mathcal{T}\{g, m\}, \tag{3}$$

where m is a physical observable, $\mathrm{G}(g)$ is a transformation group with parametrization by g, and parametric set $\mathcal{T}\{g, m\}$ is called quantum tomogram of the state $\hat{\rho}$. From physical point of view, every element of parametric set $\mathcal{T}\{g, m\}$ is a probability of observation value m after transformation G.

In case of continuous variables, *i.e.* when $\dim \mathcal{H} = \infty$, group $\mathrm{Sp}(2n, \mathbb{R})$ of phase space symplectic transformation plays role of transformation group $\mathrm{G}(g)$. Mapping (3) at that rate reads

$$\mathcal{T}(Q, \mu, \eta) = \langle Q, \mu, \eta|\hat{\rho}|Q, \mu, \eta\rangle, \quad \hat{\rho} \in \Omega(\mathcal{H}),$$

where $|Q, \mu, \eta\rangle$ is an eigenvector of the Hermitian operator $\mu\hat{q} + \eta\hat{p}$ for the eigenvalue Q. One can see that $\mathcal{T}(Q, \mu, \eta)$ is positive and normalized on Q. This representation is directly related with star-product quantization [8] and the Weyl–Heisenberg group [26].

The BDH setup reduces to mixing on beam splitter of measurable (weak) field and strong coherent field with changing phase θ. In terms of (3) the observable is $\hat{Q} = \widehat{q}\cos\theta + \widehat{p}\sin\theta$, where angle $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ could be interpreted as rotation angle of the phase space.

In case of system with discrete variables ($\dim \mathcal{H} < \infty$) mapping (3) transforms to the following relation

$$\mathcal{T}_m(U) = \langle m|U\hat{\rho}U^\dagger|m\rangle, \quad \hat{\rho} \in \Omega(\mathcal{H}), \tag{4}$$

where normalization and positivity follows directly from definition (4)

$$\sum_m \mathcal{T}_m(U) = 1, \quad \mathcal{T}_m(U) \geq 0.$$

In case $U \in \mathrm{SU}(2)$ definition (4) reduces to general definition for spin tomograms

$$U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1.$$

Here $\alpha, \beta \in \mathbb{C}$ are the Cayley–Klein parameters. In $U \in \mathrm{SU}(2)$ case the Euler angles [6] and quaternions [7] can be used for representation of tomograms [29].

4. Tomographic approach to quantum causal analysis

Here we suggest to use an approach of quantum causal analysis to bipartite system asymmetry with respect to observable outcomes described by quantum tomography. The bipartite state tomogram reads

$$\mathcal{T}_{AB}(U_A \otimes U_B) = \left\{\mathcal{T}_{AB_{ij}}(U_A \otimes U_B)\right\},$$

and reduced tomograms have the form

$$\mathcal{T}_A(U_A) = \left\{\sum_j \mathcal{T}_{AB_{ij}}(U_A \otimes U_B)\right\},$$

$$\mathcal{T}_B(U_B) = \left\{\sum_i \mathcal{T}_{AB_{ij}}(U_A \otimes U_B)\right\}.$$