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Multi-photon Rabi model: Generalized parity and its applications

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ABSTRACT

Quantum multi-photon spin–boson model is considered. We solve an operator Riccati equation associated with that model and present a candidate for a generalized parity operator allowing to transform spin–boson Hamiltonian to a block-diagonal form what indicates an existence of the related symmetry of the model.

1. Introduction

Phenomenological modeling of interacting matter and (quantized) light in physics has a long and interesting history [1]. It is of quantum optical origin [2] but is present in wide range of other branches of physics such as condensed matter [3–6] or involving mechanical oscillators [7,8]. The Rabi model [9,10], describing a qubit coupled to a single-mode electromagnetic field, is the one which has attracted continuous attention for almost a century. More [11–14] and less [15,16] recent studies on its integrability have inspired increasingly growing research.

An existence of a symmetry of any quantum model is directly related to a quality of our understanding of its properties [17]. A 'sufficient' (in certain sense) symmetry can result in an integrability of the model [18]. That is why seeking for any underlying symmetry of quantum models is always of great interest and often of great importance. In this Letter we present our contribution to this activity. We consider a family of generalized single-mode Rabi models [19]:

$$\mathrm{H} = \alpha\sigma_x + \omega a^\dagger a + \sigma_z\big(g^* a^k + g\big(a^\dagger\big)^k\big), \tag{1}$$

where σ_z and σ_x are the Pauli matrices, α and ω correspond to the energy gap of the spin and boson, respectively, whereas a and $a^\dagger$ are the annihilation and creation operators of quantized mode of light satisfying canonical commutation relation, $[a, a^\dagger] = \mathbb{I}$. It is assumed that the coupling between the qubit and the field, controlled by the strength constant g, incorporates $k > 0$ photons.

In this Letter, by solving an operator Riccati equation associated with Eq. (1), we construct an operator exhibiting significant similarities to the parity operator acting on the bosonic space. This operator, the generalized parity, can be used to simplify multi-photon Rabi model (1) and transform it to a block-diagonal form. Our work is a complementary expansion of certain results obtained in Ref. [19] for $k = 1$ and $k = 2$ in the context of approximate methods of solving the Rabi model.

The Letter is organized as follows: In Section 2 we present operator Riccati equation associated with (1) serving as a main tool applied in our studies. Next, in Section 3 the known results concerning the $k = 1, 2$ cases are reviewed. Section 4 has been devoted to the construction of the generalized parity and contains main results of our work. Finally, in Section 5, followed by conclusions, we apply the general construction to a simple example.

2. A tool: Riccati equation

Multi-photon Rabi model considered here belongs to a general class of qubit–environment composite systems described by Hamiltonian

$$\mathrm{H}_{\mathrm{QE}} = \mathrm{H}_{\mathrm{Q}} \otimes \mathbb{I}_{\mathrm{E}} + \mathbb{I}_{\mathrm{Q}} \otimes \mathrm{H}_{\mathrm{E}} + \mathrm{H}_{\mathrm{int}} \sim \begin{bmatrix} \mathrm{H}_+ & \mathrm{V} \\ \mathrm{V}^\dagger & \mathrm{H}_- \end{bmatrix} \equiv \mathbf{H}_{\mathrm{QE}}, \tag{2}$$

where H_{Q} (H_{E}) is the Hamiltonian of the system (environment). $\mathrm{H}_{\mathrm{int}}$ is the interaction of the qubit with its surroundings. $\mathbb{I}_{\mathrm{Q}}$ and $\mathbb{I}_{\mathrm{E}}$ are identities acting on corresponding Hilbert spaces $\mathbb{C}^2$ and $\mathcal{H}_{\mathrm{E}}$. The total Hamiltonian H_{QE} acts on $\mathbb{C}^2 \otimes \mathcal{H}_{\mathrm{E}}$ and the symbol $\sim$

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should be understand as '*it corresponds to*' in the sense of block operator matrix representation of operators. This correspondence is established via the isomorphism $\mathbb{C}^2 \otimes \mathcal{H}_E \sim \mathcal{H}_E \oplus \mathcal{H}_E$. Finally, the form of remaining operators $H_\pm$ and V depends upon how H_Q, H_E and H_{int} are defined.

Any steps toward diagonalization of $\mathbf{H}_{QE}$ is valuable as it can be followed by variety of different approximation schemes [19]. There is often an additional benefit emerging form such transformations which can help to exhibit useful symmetry properties being often obscured by an 'improper choice' of a basis. As it is pointed out below it is also the case of the multi-photon Rabi model (1) discussed in this Letter.

Our idea originates from an observation that Hamiltonian $\mathbf{H}_{QE}$ can be converted to a block-diagonal form

$$\mathbf{S}^{-1}\mathbf{H}_{QE}\mathbf{S} = \begin{bmatrix} H_+ + VX & 0 \\ 0 & H_- - (VX)^\dagger \end{bmatrix}, \quad \text{with } \mathbf{S} = \begin{bmatrix} \mathbb{I}_E & -X^\dagger \\ X & \mathbb{I}_E \end{bmatrix}, \tag{3}$$

provided that X satisfies an operator Riccati equation

$$XVX + XH_+ - H_-X - V^\dagger = 0. \tag{4}$$

For general considerations regarding an operator Riccati equation we refer the reader to [20–22]. This equation provides valuable tool allowing to study the exact diagonalization [23,24], stationary states [25] and in general, the dynamics [26] of two level open quantum systems [27,28]. From the decomposition (3) it is evident that the dynamics of a qubit–environment quantum system is actually governed by the Riccati (4) and pair of *uncoupled* Schrödinger equations.

For the k-photon Rabi model studied in our Letter

$$H_\pm = \omega a^\dagger a \pm \left(g^* a^k + g\left(a^\dagger\right)^k\right), \qquad V = \alpha \mathbb{I}_{\mathcal{H}_B} \tag{5}$$

and the corresponding Riccati equation reads as follows:

$$\alpha X^2 + XH_+ - H_-X - \alpha = 0. \tag{6}$$

Its mathematical properties has already been addressed in literature [29,30]. In Eq. (6), $H_\pm$ are operators acting on the bosonic Fock space $\mathcal{H}_B$, α is a real constant, whereas X is a solution to be found. If it does not lead to a confusion, we write α rather than $\alpha\mathbb{I}_{\mathcal{H}_B}$, with $\mathbb{I}_{\mathcal{H}_B}$ being the identity on $\mathcal{H}_B$.

3. Known solutions: $k = 1, 2$

For the sake of self-consistency, we begin with reviewing known solutions and their properties for the two particular cases, where $k = 1, 2$. For the simplest possible case, $k = 1$ the solution of the Riccati equation (6) was found in [24] to be the bosonic parity operator

$$P = \sum_{n\in\mathbb{N}} e^{i\pi n} |n\rangle\langle n| = \sum_{n\in\mathbb{N}} (-1)^n |n\rangle\langle n|, \tag{7}$$

which can also be written in a more compact form as $P = \exp(i\pi a^\dagger a)$, where $\{|n\rangle\}_{n\in\mathbb{N}}$ is the Fock basis, i.e., $a^\dagger a|n\rangle = n|n\rangle$. Such operator is both hermitian and unitary, hence it is an involution ($P^2 = \mathbb{I}_{\mathcal{H}_B}$). Interestingly, it solves Eq. (6) for both $\alpha = 0$ (dephasing [31,32]) and $\alpha \neq 0$ (exchange energy between the systems is present) cases, although they reflect quite different physical processes.

In the context of RWA-type approximation methods the two-photon Rabi model was studied in details within [19]. The two-photon parity operator

$$T := \exp\left[i\frac{\pi}{2} a^\dagger a\left(a^\dagger a - 1\right)\right], \tag{8}$$

was introduced therein. It has not been stated explicitly in [19] but the parity T is, as will be shown below, a solution of the Riccati equation (6) for $k = 2$.

Note if a solution such that $X^2 = 1$ exists, it allows for a symmetry (constant of motion) to be easily found. Indeed, for $J = \sigma_x \otimes X$ we have $[H, X] = 0$ as one can verify directly. Lets put it differently: If a parity operator solves the Riccati equation (6) then the Rabi model has a parity symmetry. Obviously this holds for $k = 1, 2$; in fact this is true for all k as we will shortly see.

Interestingly, two-photon Rabi model has also $\mathbb{Z}_4$ symmetry, $I_2 \otimes \sqrt{P}$. This additional constant of motion can be exploited to get the exact solution of this model [33]. On the other hand, we can write $\sigma_x \otimes \sqrt[k]{P}$ for two cases $k = 1, 2$ simultaneously. This suggest that if we set $J_k = \sigma_x \otimes \sqrt[k]{P}$ then perhaps $[H, J_k] = 0$ holds for every k. Unfortunately, this is not true because of the relation

$$\begin{aligned} \sqrt[k]{P}^\dagger a \sqrt[k]{P} &= e^{-i\frac{\pi}{k}a^\dagger a} a e^{i\frac{\pi}{k}a^\dagger a} \\ &= a + \frac{1}{1!}\cdot\frac{\pi}{k}\left[a^\dagger a, a\right] + \frac{1}{2!}\cdot\frac{\pi^2}{k^2}\left[a^\dagger a, \left[a^\dagger a, a\right]\right] + \cdots \\ &= e^{i\frac{\pi}{k}} a \end{aligned} \tag{9}$$

which shows that after transformation $a \to c_k a$ we have $c_k^2 = 1$ only for $k = 1, 2$. Note also that $\sqrt[k]{P}$ is not a solution of (6) except $k = 1$.

4. General case: $k > 0$

In what follows we show how to construct a solution of the Riccati equation (6) with coefficients $H_\pm$ provided by (5) in the general case $k > 0$. Before we start let us emphasize that the parity operator P (T) introduced in the preceding section solves Eq. (6) not only for $k = 1$ ($k = 2$) but also for all odd $k = 2n + 1$ (even, of the form $k = 4n + 2$) cases. This has already been noticed in [19]. Here we will not only fill the remaining gap $k = 4n + 4$ but also present unified approach allowing to obtain a linear solution for arbitrary k. As a first step toward constructing this solution, we define a family of orthogonal projectors

$$P_l := \sum_{n=0}^{\infty} |n, l\rangle\langle n, l|, \quad \text{with } |n, l\rangle := |kn + l - 1\rangle, \tag{10}$$

for $n \in \mathbb{N}$ and $1 \leqslant l \leqslant k$. The states $|n, l\rangle$ satisfy the following orthogonality condition:

$$\langle i, n | j, m\rangle = \delta_{kn+i-1, km+j-1} = \delta_{ij}\delta_{nm}, \tag{11}$$

where δ_{xy} is the Kronecker delta. The first equality in Eq. (11) comes from the orthogonality of the Fock basis. The second one can be justified as follows.

When $i = j$ both sides of (11) reduce to δ_{nm} since $\delta_{kn+i-1,km+i-1} = \delta_{nm}$. If $i \neq j$ (say $i > j$) the right hand side is zero. The left hand side also vanishes, as one gets either $m = n$ or $m \neq n$ in this case. Indeed, if $m = n$ then, to get nonzero left hand side, one would expect $i - j = 0$ what is impossible. Finally, for $m \neq n$ ($m > n$, say) one would expect that $k(m - n) = j - i$ (in order to keep the left side nonzero) what also does not occur as $k > i - j$ and it cannot divide $i - j$.

For a given family of orthogonal projectors one can split the space $\mathcal{H}_B$ into k subspaces so that

$$\mathcal{H}_B = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_{k-1} \oplus \mathcal{H}_k = \bigoplus_{l=1}^{k} \mathcal{H}_l, \tag{12}$$

where $\mathcal{H}_l := P_l(\mathcal{H}_B)$. The symbol $\oplus$ indicates the (orthogonal) direct sum of Hilbert spaces. Hereafter, we use it interchangeably with + when it refers to the sum of operators.