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Cantor-type cylindrical-coordinate method for differential equations with local fractional derivatives

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ABSTRACT

In this Letter, we propose to use the Cantor-type cylindrical-coordinate method in order to investigate a family of local fractional differential operators on Cantor sets. Some testing examples are given to illustrate the capability of the proposed method for the heat-conduction equation on a Cantor set and the damped wave equation in fractal strings. It is seen to be a powerful tool to convert differential equations on Cantor sets from Cantorian-coordinate systems to Cantor-type cylindrical-coordinate systems.

1. Introduction

Everything in the Cartesian-coordinate system is used to be measured with respect to the coordinate axes. However, the Cartesian coordinates are not best suited for every shape. Certain shapes, like circular or spherical ones, cannot even be demonstrated through a function in Cartesian-coordinate system. These shapes are more easily determined in cylindrical or spherical coordinates [1]. They are the equivalent of the origin in the Cartesian-coordinate system. Both classical and fractional differential equations in the coordinate system are switched between Cartesian, cylindrical and spherical coordinates [2,3].

Recently, the Cantorian-coordinate system, which was first construed in [4–6], was set up on fractals. Based on it, the heat-conduction equation on Cantor sets without heat generation in fractal media was presented in [4] as follows:

$$K^{2\alpha}\nabla^{2\alpha}T-\rho_\alpha c_\alpha\frac{\partial^\alpha T}{\partial t^\alpha}=0 \tag{1}$$

or

$$K^{2\alpha}\left(\frac{\partial^{2\alpha}T}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha}T}{\partial y^{2\alpha}}+\frac{\partial^{2\alpha}T}{\partial z^{2\alpha}}\right)-\rho_\alpha c_\alpha\frac{\partial^\alpha T}{\partial t^\alpha}=0, \tag{2}$$

where

$$\nabla^{2\alpha}=\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha}}{\partial y^{2\alpha}}+\frac{\partial^{2\alpha}}{\partial z^{2\alpha}}$$

is the local fractional Laplace operator [4–6], whose local fractional differential operator is denoted as follows [4–11] (for other definitions, see also [12–25]):

$$f^{(\alpha)}(x_0)=\left.\frac{d^\alpha f(x)}{dx^\alpha}\right|_{x=x_0}=\lim_{x\to x_0}\frac{\Delta^\alpha(f(x)-f(x_0))}{(x-x_0)^\alpha}, \tag{3}$$

where $\Delta^\alpha(f(x)-f(x_0))\cong\Gamma(1+\alpha)\Delta(f(x)-f(x_0))$ and $f(x)$ is satisfied with the following condition [4,15]:

$$\left|f(x)-f(x_0)\right|\leqslant\tau^\alpha|x-x_0|^\alpha,$$

so that (see [4–18])

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$$\left|f(x)-f(x_0)\right|<\varepsilon^{\alpha} \tag{4}$$

with $|x-x_0|<\delta$, for $\varepsilon,\delta>0$ and $\varepsilon,\delta\in\mathrm{R}$.

In a similar manner, for a given vector function $\mathbf{F}(t)=F_1(t)\mathbf{e}_1^{\alpha}+F_2(t)\mathbf{e}_2^{\alpha}+F_3(t)\mathbf{e}_3^{\alpha}$, the local fractional vector derivative is defined by (see [4])

$$\mathbf{F}^{(\alpha)}(t_0)=\left.\frac{d^{\alpha}\mathbf{F}(t)}{dt^{\alpha}}\right|_{t=t_0}=\lim_{t\to t_0}\frac{\Delta^{\alpha}(\mathbf{F}(t)-\mathbf{F}(t_0))}{(t-t_0)^{\alpha}} \tag{5}$$

where $\mathbf{e}_1^{\alpha}$, $\mathbf{e}_2^{\alpha}$ and $\mathbf{e}_2^{\alpha}$ are the directions of the local fractional vector function.

The aim of this Letter is to investigate the Cantor-type cylindrical-coordinate method within the local fractional vector operator. The layout of the Letter is as follows. In Section 2, we propose and describe the Cantor-type cylindrical-coordinate method. In Section 3, we consider the testing examples. Finally, in Section 4, we present our concluding remarks and observations.

2. Cantor-type cylindrical-coordinate method

For the following Cantor-type cylindrical coordinates [4]:

$$\begin{cases} x^{\alpha}=R^{\alpha}\cos_{\alpha}\theta^{\alpha},\\ y^{\alpha}=R^{\alpha}\sin_{\alpha}\theta^{\alpha},\\ z^{\alpha}=z^{\alpha},\end{cases} \tag{6}$$

with $R>0$, $z\in(-\infty,+\infty)$, $0<\theta<2\pi$ and $x^{2\alpha}+y^{2\alpha}=R^{2\alpha}$, we have the local fractional vector given by

$$\mathbf{r}=R^{\alpha}\cos_{\alpha}\theta^{\alpha}\mathbf{e}_1^{\alpha}+R^{\alpha}\sin_{\alpha}\theta^{\alpha}\mathbf{e}_2^{\alpha}+z^{\alpha}\mathbf{e}_3^{\alpha}, \tag{7}$$

so that

$$\begin{cases} \mathbf{C}_R^{\alpha}=\dfrac{1}{\Gamma(1+\alpha)}\dfrac{\partial^{\alpha}\mathbf{r}}{\partial R^{\alpha}}=\cos_{\alpha}\theta^{\alpha}\mathbf{e}_1^{\alpha}+\sin_{\alpha}\theta^{\alpha}\mathbf{e}_2^{\alpha},\\ \mathbf{C}_{\theta}^{\alpha}=\dfrac{1}{\Gamma(1+\alpha)}\dfrac{\partial^{\alpha}\mathbf{r}}{\partial\theta^{\alpha}}\\ \quad=-\dfrac{R^{\alpha}}{\Gamma(1+\alpha)}\sin_{\alpha}\theta^{\alpha}\mathbf{e}_1^{\alpha}+\dfrac{R^{\alpha}}{\Gamma(1+\alpha)}\cos_{\alpha}\theta^{\alpha}\mathbf{e}_2^{\alpha},\\ \mathbf{C}_3^{\alpha}=\dfrac{1}{\Gamma(1+\alpha)}\dfrac{\partial^{\alpha}\mathbf{r}}{\partial z^{\alpha}}=\mathbf{e}_3^{\alpha}.\end{cases} \tag{8}$$

Therefore, we obtain

$$\begin{cases} \mathbf{e}_R^{\alpha}=\cos_{\alpha}\theta^{\alpha}\mathbf{e}_1^{\alpha}+\sin_{\alpha}\theta^{\alpha}\mathbf{e}_2^{\alpha},\\ \mathbf{e}_{\theta}^{\alpha}=-\sin_{\alpha}\theta^{\alpha}\mathbf{e}_1^{\alpha}+\cos_{\alpha}\theta^{\alpha}\mathbf{e}_2^{\alpha},\\ \mathbf{e}_z^{\alpha}=\mathbf{e}_3^{\alpha},\end{cases} \tag{9}$$

where $\mathbf{C}_R^{\alpha}=\mathbf{e}_R^{\alpha}$, $\mathbf{C}_{\theta}^{\alpha}=\frac{R^{\alpha}}{\Gamma(1+\alpha)}\mathbf{e}_{\theta}^{\alpha}$, $\mathbf{C}_3^{\alpha}=\mathbf{e}_z^{\alpha}$.

Now, by making use of Eq. (9), we can write this last result in matrix form as follows:

$$\begin{pmatrix}\mathbf{e}_R^{\alpha}\\ \mathbf{e}_{\theta}^{\alpha}\\ \mathbf{e}_z^{\alpha}\end{pmatrix}=\begin{pmatrix}\cos_{\alpha}\theta^{\alpha} & \sin_{\alpha}\theta^{\alpha} & 0\\ -\sin_{\alpha}\theta^{\alpha} & \cos_{\alpha}\theta^{\alpha} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\mathbf{e}_1^{\alpha}\\ \mathbf{e}_2^{\alpha}\\ \mathbf{e}_3^{\alpha}\end{pmatrix}, \tag{10}$$

which leads to

$$\mathbf{E}_i^{\alpha}=\mathbf{T}_{ij}^{\alpha}\mathbf{E}_j^{\alpha}, \tag{11}$$

where

$$\mathbf{E}_i^{\alpha}=\begin{pmatrix}\mathbf{e}_R^{\alpha}\\ \mathbf{e}_{\theta}^{\alpha}\\ \mathbf{e}_z^{\alpha}\end{pmatrix},\qquad \mathbf{T}_{ij}^{\alpha}=\begin{pmatrix}\cos_{\alpha}\theta^{\alpha} & \sin_{\alpha}\theta^{\alpha} & 0\\ -\sin_{\alpha}\theta^{\alpha} & \cos_{\alpha}\theta^{\alpha} & 0\\ 0 & 0 & 1\end{pmatrix},$$
$$\mathbf{E}_j^{\alpha}=\begin{pmatrix}\mathbf{e}_1^{\alpha}\\ \mathbf{e}_2^{\alpha}\\ \mathbf{e}_3^{\alpha}\end{pmatrix}. \tag{12}$$

Here $\mathbf{T}_{ij}^{\alpha}$ is fractal matrix, which is defined on the generalized Banach space [5,6]. The general basis vectors of two fractal spaces are defined, respectively, from the fractal tangent vectors [4], namely,

$$\mathbf{E}_i^{\alpha}=\begin{pmatrix}\mathbf{e}_R^{\alpha}\\ \mathbf{e}_{\theta}^{\alpha}\\ \mathbf{e}_z^{\alpha}\end{pmatrix},\qquad \mathbf{E}_j^{\alpha}=\begin{pmatrix}\mathbf{e}_1^{\alpha}\\ \mathbf{e}_2^{\alpha}\\ \mathbf{e}_3^{\alpha}\end{pmatrix}. \tag{13}$$

In view of Eqs. (8) and (9), upon differentiating the Cantorian position with respect to the Cantor-type cylindrical coordinates implies that

$$\begin{cases} \mathbf{e}_R^{\alpha}=\dfrac{1}{\Gamma(1+\alpha)}\dfrac{\partial^{\alpha}\mathbf{r}}{\partial R^{\alpha}}=\cos_{\alpha}\theta^{\alpha}\mathbf{e}_1^{\alpha}+\sin_{\alpha}\theta^{\alpha}\mathbf{e}_2^{\alpha},\\ \mathbf{e}_{\theta}^{\alpha}=\dfrac{1}{R^{\alpha}}\dfrac{\partial^{\alpha}\mathbf{r}}{\partial\theta^{\alpha}}=-\sin_{\alpha}\theta^{\alpha}\mathbf{e}_1^{\alpha}+\cos_{\alpha}\theta^{\alpha}\mathbf{e}_2^{\alpha},\\ \mathbf{e}_z^{\alpha}=\dfrac{1}{\Gamma(1+\alpha)}\dfrac{\partial^{\alpha}\mathbf{r}}{\partial z^{\alpha}}=\mathbf{e}_3^{\alpha}.\end{cases} \tag{14}$$

Eq. (14) is orthogonal and normalized everywhere (see [5,6]). Hence, we can define a local fractal basis with an orientation, which is derived from one fractal space to another fractal space. Based on this, a local fractional vector field can be defined as follows:

$$\mathbf{r}(R,\theta,z)=\mathbf{r}\cdot\left(\mathbf{e}_R^{\alpha}+\mathbf{e}_{\theta}^{\alpha}+\mathbf{e}_z^{\alpha}\right) \tag{15}$$

where the fractal vector coordinates given by

$$r_R=\mathbf{r}(R,\theta,z)\cdot\mathbf{e}_R^{\alpha},\qquad r_{\theta}=\mathbf{r}(R,\theta,z)\cdot\mathbf{e}_{\theta}^{\alpha},$$
$$r_z=\mathbf{r}(R,\theta,z)\cdot\mathbf{e}_z^{\alpha} \tag{16}$$

are the projections of $\mathbf{r}$ on the local fractal basis vectors.

The local fractional derivatives with respect to the Cantor-type cylindrical coordinates are given by the local fractional differentiation through the Cantorian coordinates as follows:

$$\begin{aligned}\frac{\partial^{\alpha}}{\partial R^{\alpha}}&=\left(\frac{\partial x}{\partial R}\right)^{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}+\left(\frac{\partial y}{\partial R}\right)^{\alpha}\frac{\partial^{\alpha}}{\partial y^{\alpha}}+\left(\frac{\partial z}{\partial R}\right)^{\alpha}\frac{\partial^{\alpha}}{\partial z^{\alpha}}\\ &=\Gamma(1+\alpha)\left(\cos_{\alpha}\theta^{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}+\sin_{\alpha}\theta^{\alpha}\frac{\partial^{\alpha}}{\partial y^{\alpha}}\right)\\ &=\mathbf{e}_R^{\alpha}\cdot\nabla^{\alpha}=\nabla_R^{\alpha},\end{aligned} \tag{17}$$

$$\begin{aligned}\frac{\partial^{\alpha}}{\partial\theta^{\alpha}}&=\left(\frac{\partial x}{\partial\theta}\right)^{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}+\left(\frac{\partial y}{\partial\theta}\right)^{\alpha}\frac{\partial^{\alpha}}{\partial y^{\alpha}}+\left(\frac{\partial z}{\partial\theta}\right)^{\alpha}\frac{\partial^{\alpha}}{\partial z^{\alpha}}\\ &=R^{\alpha}\left(-\sin_{\alpha}\theta^{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}+\cos_{\alpha}\theta^{\alpha}\frac{\partial^{\alpha}}{\partial y^{\alpha}}\right)\\ &=R^{\alpha}\mathbf{e}_{\theta}^{\alpha}\cdot\nabla^{\alpha}=R^{\alpha}\nabla_{\theta}^{\alpha}\end{aligned} \tag{18}$$

and

$$\begin{aligned}\frac{\partial^{\alpha}}{\partial z^{\alpha}}&=\left(\frac{\partial x}{\partial z}\right)^{\alpha}\frac{\partial^{\alpha}}{\partial x^{\alpha}}+\left(\frac{\partial y}{\partial z}\right)^{\alpha}\frac{\partial^{\alpha}}{\partial y^{\alpha}}+\left(\frac{\partial z}{\partial z}\right)^{\alpha}\frac{\partial^{\alpha}}{\partial z^{\alpha}}\\ &=\Gamma(1+\alpha)\frac{\partial^{\alpha}}{\partial z^{\alpha}}=\mathbf{e}_z^{\alpha}\cdot\nabla^{\alpha}=\nabla_z^{\alpha},\end{aligned} \tag{19}$$

where

$$\nabla_R^{\alpha}=\mathbf{e}_R^{\alpha}\cdot\nabla^{\alpha}=\frac{\partial^{\alpha}}{\partial R^{\alpha}},\qquad \nabla_{\theta}^{\alpha}=R^{\alpha}\mathbf{e}_{\theta}^{\alpha}\cdot\nabla^{\alpha}=\frac{1}{R^{\alpha}}\frac{\partial^{\alpha}}{\partial\theta^{\alpha}},$$
$$\frac{\partial^{\alpha}}{\partial z^{\alpha}}=\mathbf{e}_z^{\alpha}\cdot\nabla^{\alpha}=\nabla_z^{\alpha}. \tag{20}$$

In light of Eq. (20), the local fractional gradient operator is described as follows:

$$\begin{aligned}\nabla^{\alpha}&=\mathbf{e}_R^{\alpha}\nabla_R^{\alpha}+\mathbf{e}_{\theta}^{\alpha}\nabla_{\theta}^{\alpha}+\mathbf{e}_z^{\alpha}\nabla_z^{\alpha}\\ &=\mathbf{e}_R^{\alpha}\frac{\partial^{\alpha}}{\partial R^{\alpha}}+\mathbf{e}_{\theta}^{\alpha}\frac{1}{R^{\alpha}}\frac{\partial^{\alpha}}{\partial\theta^{\alpha}}+\mathbf{e}_z^{\alpha}\frac{\partial^{\alpha}}{\partial z^{\alpha}}.\end{aligned} \tag{21}$$