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Multi-armed spirals and multi-pairs antispirals in spatial rock–paper–scissors games

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ABSTRACT

We study the formation of multi-armed spirals and multi-pairs antispirals in spatial rock–paper–scissors games with mobile individuals. We discover a set of seed distributions of species, which is able to produce multi-armed spirals and multi-pairs antispirals with a finite number of arms and pairs based on stochastic processes. The joint spiral waves are also predicted by a theoretical model based on partial differential equations associated with specific initial conditions. The spatial entropy of patterns is introduced to differentiate the multi-armed spirals and multi-pairs antispirals. For the given mobility, the spatial entropy of multi-armed spirals is higher than that of single armed spirals. The stability of the waves is explored with respect to individual mobility. Particularly, we find that both two armed spirals and one pair antispirals transform to the single armed spirals. Furthermore, multi-armed spirals and multi-pairs antispirals are relatively stable for intermediate mobility. The joint spirals with lower numbers of arms and pairs are relatively more stable than those with higher numbers of arms and pairs. In addition, comparing to large amount of previous work, we employ the no flux boundary conditions which enables quantitative studies of pattern formation and stability in the system of stochastic interactions in the absence of excitable media.

Formation of self-organized pattern is a fundamental aspect of physical and biological systems out of equilibrium. Spiral waves are quite common in a variety of excitable systems and population dynamics, such as Belousov–Zhabotinsky reaction [1,2], the cardiac tissue [3], inset population dynamics [4] and cyclically competing populations with mobility [5]. Spiral waves play significant roles in the dynamics of excitable systems, e.g., in heart disease, such as arrhythmia and fibrillation, which lead to death [3,6,7]. Spiral waves are important in population dynamics as well. In particular, biodiversity in cyclically competing populations with stochastic interactions can be maintained and stabilized by entangled moving spiral waves [5,8]. The coexistence of two or more spirals may form multi-armed spiral and antispiral waves. These interesting joint spirals have been extensively studied in excitable systems theoretically and experimentally [9–14]. However, in the population dynamics in the presence of stochastic processes, multi-armed spirals and multi-pairs antispirals among entangled spirals is rarely studied and far from being well understood. There are two important open questions associated with these waves: Are they able to be generated through stochastic interactions and how is their stability? The purpose of this Letter is to address these questions in the framework of cyclic competing games with mobile individuals.

Non-hierarchical cyclic competitions have been observed in a number of real ecosystems, ranging from colicinogenic microbes competition to mating strategies of side-blotched lizards in California [15–19], as well as human sociality in terms of public goods games [20–22]. The essential features of such competition can be captured by the childhood game “rock–paper–scissors” (RPS). In the game, species coexistence, as the key factor for maintaining biodiversity, has been given much attention, especially for the conditions that ensure species coexistence [23–31]. Both laboratory experiment and theoretical model have revealed that spatial structure by confining local interaction is necessary for stabilizing species coexistence [19]. Otherwise, stochastic effect and external perturbation can easily ruin biodiversity. Quite recently, individual mobility has been incorporated in the spatial RPS game [5,8,32,33]. It has been found that individual mobility induces entangled moving spiral waves which preclude species from extinction [5]. The stochastic game has been casted into a set of partial differential equations by a continuous approximation [8]. In this Letter, we investigate the origin of multi-armed spiral waves and multi-pairs

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antispiral waves on the basis of the spatial RPS game with mobile individuals, which is unaddressed prior to our work. We find that the joint spiral waves can spontaneously arise due to the interaction of neighboring spirals and the type of the joint spirals is determined by the position and rotational directions of neighboring spirals. In particular, we discover a general set of seeds of species distribution, which is capable of producing multi-armed spirals with a finite number of arms and antispirals with finite number of pairs. The diverse patterns generated from stochastic simulations are reproduced by solving a set of partial differential equations from specific initial conditions. We have also discussed the stability of the joint spiral waves with respect to individual mobility.

We consider the spatial RPS game proposed in Ref. [5]. Nodes of a $L \times L$ square lattice with no flux boundary conditions sustain mobile individuals belonging to one of the three species, A, B and C. Each node can either host one individual of a given species or it can be vacant. Vacant sites, denoted by $\otimes$, are also the so-called resource sites where individuals of species reproduce offspring. The dynamical process can be described as following:

$$AB \xrightarrow{1} A\otimes, \qquad BC \xrightarrow{1} B\otimes, \qquad CA \xrightarrow{1} C\otimes, \tag{1}$$

$$A\otimes \xrightarrow{1} AA, \qquad B\otimes \xrightarrow{1} BB, \qquad C\otimes \xrightarrow{1} CC, \tag{2}$$

$$A\odot \xrightarrow{\gamma} \odot A, \qquad B\odot \xrightarrow{\gamma} \odot B, \qquad C\odot \xrightarrow{\gamma} \odot C \tag{3}$$

where $\odot$ denotes any species or vacant sites. These reactions describe three processes, i.e. competition, reproduction and exchange, occurring only between neighboring nodes. In reaction (1), species A eliminates species B at a rate 1, whereby the node previously hosting species B becomes vacant. In the same manner species B can kill species C, and species C can kill species A, thus forming a closed loop. In reaction (2), individuals place an offspring to a neighboring vacant node $\otimes$ at a rate 1. Reaction (3) defines exchange process where an individual exchanges its position with an individual belonging any species or an empty site at a rate γ. According to the theory of random walks [34], mobility of individuals M is defined as: $M = \gamma/2N$, where $N = L \times L$ and M represents the typical area explored by one mobile individual per unit time.

We apply stochastic algorithm developed by Gillespie to simulate the system's evolution [35], where the occurring probabilities of reactions are determined by their rates. In our model, competition and reproduction occur with probability $1/(\gamma + 2)$, whereas exchange (moving) occurs with probability $\gamma/(\gamma + 2)$. At each step, an individual is randomly selected to interact with one randomly selected neighboring site. In one time step, all individuals are selected once on average.

A critical value $M_c = (4.5 \pm 0.5) \times 10^{-4}$ of mobility has been identified in Ref. [5]. Below M_c, three subpopulations can stably coexist in the form of moving spiral waves; while above M_c, the wave length of spirals exceeds the size of underlying lattice and biodiversity is lost. Here, we focus on the biodiversity region for $M < M_c$. In this region, by carrying out sufficient stochastic simulations from random initial distributions of species, we found there is chance to observe both multi-armed spirals and multi-pairs antispirals, as shown in Fig. 1(b) and (e). For different specific initial conditions (see Fig. 2(b) and (c) for details), a two-armed spiral and an one-pair antispiral can be reproduced, as shown in Fig. 1(a) and (c) respectively, which are qualitatively the same as the marked patterns in Fig. 1(b). In addition, as shown in Fig. 1(a) and (c) respectively, a one-armed spiral and a two-pairs antispiral emerge from special initial conditions (see Fig. 2(d) and (e) for details), which are observed in Fig. 1(e). We also found that these patterns can last for relative long time and then they may disappear or transform to single armed spirals with the initial conditions of species randomly distributing on the lattice. In the

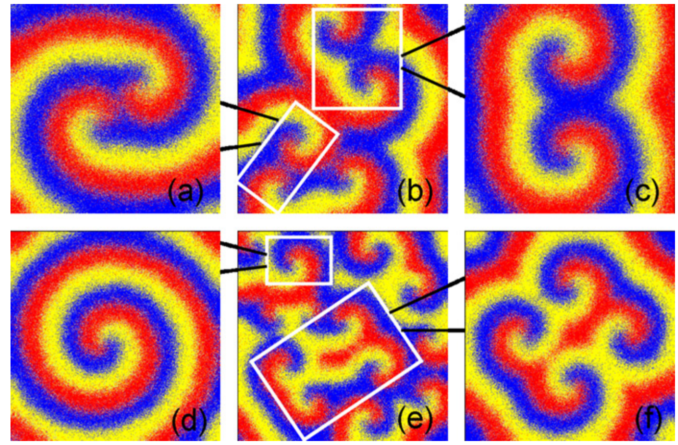

Fig. 1. (Color online.) Spatial patterns in RPS game for $M = 5.0 \times 10^{-5}$. Panels (b) and (e) are obtained from random distribution of three species initially. In panels (a), (c), (d), and (f), the system starts from specific seed distributions of three species. The marked local patterns in (b) can be reproduced from specific initial conditions, as shown in (a) and (c). The marked local patterns in (e) can be generated as well, as shown in (d) and (f). $L = 512$ for all panels.

multi-armed spirals, the arms rotate in the same direction with the same speed, resulting exclusively from stochastic interactions among neighboring individuals. In the antispirals, the two spirals of a pair rotate with the same speed but in reverse directions. The identical rotational speed of sub-spirals in the waves ensures their stable existence. It is noteworthy mentioning that the patterns in Fig. 1 are obtained from no flux boundary conditions, and we also examine the phase transition of system from biodiversity to uniformity with no flux boundary conditions. As shown in Fig. 2(a), a critical mobility M_c emerges at 4.5×10^{-4}, which is the same as the result of periodic boundary conditions in Ref. [5].

It is interesting to find that the multi-armed spirals and multi-pairs antispirals can arise from some specific distribution of three subpopulations. As shown in Figs. 2(b) and 2(c), square, triangle and circle symbols stand for a small amount of three subpopulations which are placed on a lattice with no flux boundary condition. Other sites of the lattice are left empty. In the early stage, each pile of individuals expand due to reproduction. After the boundaries of different species encounter, populations begin to rotate because of the cyclic competition. Finally, after the systems reaching a non-equilibrium steady state, a two-armed spiral and a one-pair antispirals emerge. Let's see Fig. 2(b), the six pile of species placed around a circle are in the order A, B, C, A, B, and C. The six piles can be separated into two groups, each of which contains three species. During the evolution, each group form an arm. Due to the spatial symmetry of the two group, the wave length, rotation speeds and directions of the two arms are the same, giving rise to a steady two-armed spiral (Fig. 1(a)). In contrast, to generate antispirals, we need to place a finite number of species at the center of a circle and the other two species around the circle (Fig. 2(c)), leading to a steady one-pair antispiral (Fig. 1(c)).

By extending the simple configuration in Fig. 2(b) and (c), we discover a general route to generate multi-armed spirals with a finite number of arms and antispirals with a finite number of pairs. To articulate the method, we should define the basic cell in the initial distribution of species. As shown in Fig. 2(d), the cell of multi-armed spirals is composed of three species in the order A, B and C. The cell of antispiral contains two species except the central species. The central species can be a finite number, but once the central species is fixed, the cell is fixed as well. For the multi-armed spirals, the number of arms is determined by the number of cells. In general, one arm can be formed by one cell, so that by adjusting the number of cells, one can obtain multi-armed spirals