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Compacton and solitary pattern solutions for nonlinear dispersive KdV-type equations involving Jumarie's fractional derivative ☆

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ABSTRACT

In this Letter, the fractional variational iteration method using He's polynomials is implemented to construct compacton solutions and solitary pattern solutions of nonlinear time-fractional dispersive KdV-type equations involving Jumarie's modified Riemann–Liouville derivative. The method yields solutions in the forms of convergent series with easily calculable terms. The obtained results show that the considered method is quite effective, promising and convenient for solving fractional nonlinear dispersive equations. It is found that the time-fractional parameter significantly changes the soliton amplitude of the solitary waves.

1. Introduction

Recently, various nonlinear dispersive KdV-type equations have arisen in a large range of physical phenomena. They can be used to study shallow-water waves [1], optical solitons in the two-cycle regime [2], density waves in traffic flow of two kinds of vehicles [3], short waves in nonlinear dispersive models [4], surface acoustic soliton in a system supporting Love waves [5] and so on. Due to the importance of the KdV-type equations, compactons, solitons and periodic solutions of these equations including $K(m,n)$ equations, $K(m,p,1)$ equations, coupled Hirota–Satsuma KdV equations and variable-coefficient mKdV equation, have been extensively investigated by many researchers [6–14].

Most recently, fractional differential equations (FDEs) have gained much attention due to the exact description of nonlinear phenomena in fluid flow, biology, physics, engineering and other areas of science. That is because of the fact that, the next state of a real physical phenomenon might depend on not only its current state but also upon its historical states(non-local property), which can be successfully modeled by using the theory of derivatives and integrals of fractional order. A natural question is that: How can we get the exact solutions of FDEs? The aim of the present Letter is to construct exact solutions of three time-fractional nonlinear dispersive KdV-type equations given in the following models:

- A variant of the time-fractional KdV equation:

$$D_t^{\alpha} u - a\big(u^2\big)_x + \big[u(u)_{xx}\big]_x + \big[u(u)_x\big]_{xx} = 0, \quad 0 < \alpha \leqslant 1. \tag{1}$$

- The time-fractional $K(2,4,-2)$ equation:

$$D_t^{\alpha} u + a\big(u^2\big)_x + b\big[u^4\big(u^{-2}\big)_{xx}\big]_x = 0, \quad 0 < \alpha \leqslant 1. \tag{2}$$

- The time-fractional $K(3,3,1)$ equation:

$$D_t^{\alpha} u + \big(u^3\big)_x - \big(u^3\big)_{xxx} + u_{xxxxx} = 0, \quad 0 < \alpha \leqslant 1. \tag{3}$$

Here a, b are real constants, $[u(u)_{xx}]_x$ and $[u(u)_x]_{xx}$ in Eq. (1), $b[u^4(u^{-2})_{xx}]_x$ in Eq. (2) and u_{xxxxx} in Eq. (3) are dispersive terms. $D_t^{\alpha}(\cdot)$ is Jumarie's modified Riemann–Liouville derivative of order α [15–17] defined in Section 2. The Jumarie's modified Riemann–Liouville derivative has many interesting properties. For example, the α-order derivative of a constant is zero, and it can be applied to functions which are differentiable or not [17]. In the following we shall use at will and for convenience, the notations $D_t^{\alpha} f(t) = f^{(\alpha)}(t)$ for the fractional derivative. As the variants of $K(m,n)$ equation, Eq. (1) and Eq. (2) emerge in nonlinear lattices [18] and can be used to describe the motion of the diluted suspension [19]. The fifth-order KdV-like equation (3) can be applied

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to study the motions of long waves in shallow water under gravity and it also appears in the theory of quantum mechanics and magneto-acoustic waves in plasma physics [20].

It is necessary to point out that, changing the integer order time derivative into a fractional one in KdV-type equations (1)–(3) has great physical impact. That's because the integer order KdV-type equations only consider the instant of time (local property), which means the obtained soliton solutions may not confirm the solitary waves in the real world very well. However, by introducing and modulating the time-fractional derivative parameter α, this problem can be overcome. For example, when studying the electron-acoustic waves in unmagnetized plasmas, the researchers introduced the integer order KdV equation. However, the obtained soliton solution underestimates the amplitude of the solitary waves by more than 20%. To increase the amplitude, Prof. S.A. El-Wakil et al. converted the integer order KdV equation into a time-fractional one [21] and the obtained results were in agreement with some observations from the Viking satellite.

For better understanding the mechanisms of the complicated nonlinear physical phenomena as well as further applying them in practical life, searching for explicit solutions of the aforementioned three nonlinear time-fractional dispersive equations is of great importance. In the past, many powerful methods have been established and developed to obtain numerical and analytical solutions of FDEs, such as finite difference method [22], Adomian decomposition method [23], differential transform method [24], fractional sub-ODE method [25], fractional variational iteration method (FVIM) [26] and so on. Thanks to the efforts of many researchers, several FDEs have been investigated, such as the impulsive fractional differential equations [27], space- and time-fractional advection–dispersion equation [28], fractional generalized Burgers' fluid [29], fractional heat- and wave-like equations [30], etc.

In this Letter, we apply the fractional variational iteration method using He's polynomials (FVIMHP) [31] proposed recently for solving Eqs. (1)–(3). As the modification of FVIM, the main advantage of FVIMHP is that He's homotopy perturbation (He's polynomials) is introduced in the correct functional [32,33]. The use of Lagrange multipliers and homotopy perturbation reduces the successive application of the integral operator and the cumbersome huge computational work while still maintaining a very high level of accuracy. Therefore, the FVIMHP needs less computing time than FVIM. It is easy to see that the FVIMHP is formulated by taking the full advantages of variational iteration method (VIM) [34,35], homotopy perturbation method (HPM) [36–38] and Jumarie's modified Riemann–Liouville derivative.

This Letter is organized as follows: In Section 2, some basic definitions of fractional calculus and the main steps of FVIMHP are given. In Section 3, we construct the compacton solutions and solitary pattern solutions of Eqs. (1)–(3) via the FVIMHP. In Section 4, some conclusions are given.

2. Fractional calculus and FVIMHP

Let $f: R \to R, t \to f(t)$ denote a α-th continuous (but not necessarily differentiable) function [39], $0 < \alpha \leqslant 1$. Yang's local fractional integral [40,41] in the interval $[a,b]$ is defined as

$$ {}_aI_b^{\alpha} f(t) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^{\alpha}. \tag{4}$$

The Jumarie's modified Riemann–Liouville derivative of order α is defined by the expression [16]

$$ f^{(\alpha)}(t) = \frac{1}{\Gamma(-\alpha)} \int_0^t (t-\xi)^{-\alpha-1}\big(f(\xi)-f(0)\big)\,d\xi, \quad \alpha < 0. \tag{5}$$

For positive α, we set

$$ f^{(\alpha)}(t) = \big(f^{(\alpha-1)}(t)\big)' = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha}\big(f(\xi)-f(0)\big)\,d\xi, \quad 0 < \alpha < 1, \tag{6}$$

and

$$ f^{(\alpha)}(t) = \big(f^{(n)}(t)\big)^{(\alpha-n)}, \quad n \leqslant \alpha < n+1, \quad n \geqslant 1. \tag{7}$$

The integral with respect to $(dt)^{\alpha}$ is defined as the solution of the fractional differential equation

$$ dy = f(t)(dt)^{\alpha}, \qquad x \geqslant 0, \quad y(0) = 0, \quad 0 < \alpha \leqslant 1, \tag{8}$$

which is provided by the following result [17].

Let $f(t)$ denote a continuous function, then the solution of the Eq. (8) is defined by the equality

$$ y = \int_0^t f(\xi)(d\xi)^{\alpha} = \alpha \int_0^t (t-\xi)^{\alpha-1} f(\xi)\,d\xi, \quad 0 < \alpha \leqslant 1. \tag{9}$$

We present the essential steps of the fractional variational iteration method using He's polynomials as follows:

Step 1: Suppose that a nonlinear equation, say in two independent variables x and t, is given by

$$ D_t^{\gamma} u(x,t) = L\big(u(x,t)\big) + N\big(u(x,t)\big) + g(x,t), \tag{10}$$

where $D_t^{\gamma}(\cdot)$ is the modified Riemann–Liouville derivative, $\gamma > 0$, L is a linear operator, N is a nonlinear operator, $u = u(x,t)$ is an unknown function, and $g(x,t)$ is the forcing term.

Step 2: According to the FVIM, we can construct the following correct functional

$$ u_{k+1}(x,t) = u_k(x,t) + {}_0I_t^{\gamma}\big\{\lambda(\tau)\big(D_\tau^{\gamma} u_k(x,\tau) - L\big(u_k(x,\tau)\big) - N\big(\tilde{u}_k(x,\tau)\big) - g(x,\tau)\big)\big\}, \tag{11}$$

where λ is the Lagrange multiplier, which can be identified optimally via the variational theory. The subscript $k \geqslant 0$ denotes the k-th approximation, the function $\tilde{u}_k$ is considered as a restricted variation, that is $\delta\tilde{u}_k = 0$.

Step 3: We use HPM in the correction functional (11) as follows

$$ \sum_{k=0}^{\infty} q^k u_k(x,t) = u_0(x,t) + q\left\{\sum_{k=1}^{\infty} q^k u_k(x,t) + {}_0I_t^{\gamma}\left\{\lambda(\tau)\left(\sum_{k=0}^{\infty} q^k D_\tau^{\gamma} u_k(x,\tau) - \sum_{k=0}^{\infty} q^k L\big(u_k(x,\tau)\big) - \sum_{k=0}^{\infty} q^k N\big(\tilde{u}_k(x,\tau)\big) - g(x,\tau)\right)\right\}\right\}, \tag{12}$$