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Stronger Schrödinger-like uncertainty relations

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ABSTRACT

Uncertainty relation is one of the fundamental building blocks of quantum theory. Nevertheless, the traditional uncertainty relations do not fully capture the concept of incompatible observables. Here we present a stronger Schrödinger-like uncertainty relation, which is stronger than the relation recently derived by Maccone and Pati (2014) [11]. Furthermore, we give an additive uncertainty relation which holds for three incompatible observables, which is stronger than the relation newly obtained by Kechrimparis and Weigert (2014) [12] and the simple extension of the Schrödinger uncertainty relation.

1. Introduction

Uncertainty is one of the distinct features of quantum theory. The concept of uncertainty principle was first introduced by Heisenberg [1]. The original form of uncertainty relation was derived by Kennard [2] and Weyl [3]. Indeed, the uncertainty relation is a mathematical description of trade-off relation in the measurement statistics of two incompatible observables. It refers to the preparation of the system which has intrinsic spreads in the measurement outcomes for independent measurements. Notably, it does not mean that two incompatible observables are impossible to be measured simultaneously on a quantum system [4]. The best known formula of uncertainty relation is the Heisenberg–Robertson uncertainty relation, which bounds the product of a pair of variances through the expectation value of their commutator [5]. It reads

$$\Delta A^2 \Delta B^2 \geq \left|\frac{1}{2}\langle\psi|[A,B]|\psi\rangle\right|^2, \tag{1}$$

for arbitrary observables A, B, and any state $|\psi\rangle$, where the variances of an observable X in state $|\psi\rangle$ are defined as $\Delta X^2 = \langle\psi|X^2|\psi\rangle - \langle\psi|X|\psi\rangle^2$ and the commutator is defined by $[A,B] = AB - BA$. A stronger extension of the uncertainty relation (1) was made by Schrödinger [6], namely

$$\Delta A^2 \Delta B^2 \geq \left|\frac{1}{2}\langle[A,B]\rangle\right|^2 + \left|\frac{1}{2}\langle\{A,B\}\rangle - \langle A\rangle\langle B\rangle\right|^2, \tag{2}$$

where the anti-commutator is defined by $\{A,B\} = AB + BA$, and $\langle X\rangle$ denotes the expectation value of X.

Uncertainty relations are significant in physics, e.g., quantum mechanics and quantum information [7–10]. Traditionally the uncertainty relations try to quantitatively express the impossibility of joint sharp preparation of incompatible observables. However, in practice, they do not always capture the notion of incompatible observables since they become trivial in some cases. Recently, Maccone and Pati derived two stronger uncertainty relations based on the sum of ΔA^2 and ΔB^2 [11], which to a large extent can avoid the triviality problem and provide more stringent bounds for incompatible observables on the quantum state. The first inequality is

$$\Delta A^2 + \Delta B^2 \geq \pm i\langle[A,B]\rangle + |\langle\psi|A \pm iB|\psi^\perp\rangle|^2, \tag{3}$$

where $|\psi^\perp\rangle$ is an arbitrary state orthogonal to the state $|\psi\rangle$, the sign on the right-hand side of the inequality takes $+(-)$ while $i\langle[A,B]\rangle$ is positive (negative). The second inequality is

$$\Delta A^2 + \Delta B^2 \geq \frac{1}{2}|\langle\psi^\perp_{A+B}|A+B|\psi\rangle|^2, \tag{4}$$

where $|\psi^\perp_{A+B}\rangle \propto (A + B - \langle A+B\rangle)|\psi\rangle$ is a state orthogonal to $|\psi\rangle$. Maccone and Pati also derived an amended Heisenberg–Robertson uncertainty relation, i.e.

$$\Delta A \Delta B \geq \frac{\pm i\frac{1}{2}\langle[A,B]\rangle}{1 - \frac{1}{2}|\langle\psi|\frac{A}{\Delta A} \pm i\frac{B}{\Delta B}|\psi^\perp\rangle|^2}, \tag{5}$$

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which is stronger than the Heisenberg–Robertson uncertainty relation.

Two noncommutative sharp observables as well as three pairwise noncommutative sharp observables are incompatible, whatever the state of the system might be. Recently, two Heisenberg uncertainty relations for three canonical observables were obtained by Kechrimparis and Weigert [12]. The multiplicative uncertainty relation reads

$$\Delta p \Delta q \Delta r \geq (\frac{\hbar}{\sqrt{3}})^{\frac{3}{2}} , \tag{6}$$

where the Schrödinger triple (p, q, r) satisfies the commutation relations

$$[q, p] = [p, r] = [r, q] = i\hbar \, . \tag{7}$$

Here, the observable $r = -q - p$. They also gave an additive uncertainty relation for the Schrödinger triple (q, p, r), it reads

$$\Delta p^2 + \Delta q^2 + \Delta r^2 \geq \sqrt{3}\hbar \, . \tag{8}$$

In this work, two new Schrödinger-like uncertainty relations for the sum and product of variances of two observables by extending the Schrödinger uncertainty relation (2) are obtained. An uncertainty relation for three observables will be given, which is stronger than the uncertainty relation given by Kechrimparis and Weigert, and we will exhibit its property in case of spin-1 system.

2. Schrödinger-like uncertainty relation

The first Schrödinger-like uncertainty relation reads

$$\begin{aligned}\Delta A^2 + \Delta B^2 \geq & |\langle [A, B]\rangle + \langle \{A, B\}\rangle - 2\langle A\rangle\langle B\rangle| \\ & + |\langle\psi|A - e^{i\alpha}B|\psi^{\perp}\rangle|^2 \, ,\end{aligned} \tag{9}$$

which is valid for arbitrary states $|\psi^{\perp}\rangle$ orthogonal to the state of the system $|\psi\rangle$ and stronger than Maccone and Pati's uncertainty relation (3) (Fig. 1), where α is a real constant. If $\langle\{A, B\}\rangle - 2\langle A\rangle\langle B\rangle > 0$, then $\alpha = \arctan \frac{-i\langle[A,B]\rangle}{\langle\{A,B\}\rangle - 2\langle A\rangle\langle B\rangle}$; if $\langle\{A, B\}\rangle - 2\langle A\rangle\langle B\rangle < 0$, then $\alpha = \pi + \arctan \frac{-i\langle[A,B]\rangle}{\langle\{A,B\}\rangle - 2\langle A\rangle\langle B\rangle}$; and while $\langle\{A, B\}\rangle - 2\langle A\rangle\langle B\rangle = 0$, it reduces to (3). Removing the last term of (9), it then turns into

$$\Delta A^2 + \Delta B^2 \geq |\langle [A, B]\rangle + \langle \{A, B\}\rangle - 2\langle A\rangle\langle B\rangle| \, , \tag{10}$$

which is implied by the Schrödinger uncertainty relation (2).

The second Schrödinger-like uncertainty relation is

$$\Delta A^2 \Delta B^2 \geq \frac{\left|\frac{1}{2}\langle [A, B]\rangle\right|^2 + \left|\frac{1}{2}\langle \{A, B\}\rangle - \langle A\rangle\langle B\rangle\right|^2}{(1 - \frac{1}{2}|\langle\psi|\frac{A}{\Delta A} - e^{i\alpha}\frac{B}{\Delta B}|\psi^{\perp}\rangle|^2)^2} , \tag{11}$$

which is stronger than the Schrödinger uncertainty relation (2) and reduces to (5) when $\langle\{A, B\}\rangle - 2\langle A\rangle\langle B\rangle = 0$.

Proof. To prove the uncertainty relation (9), we start by introducing a general inequality

$$\|c_A \bar{A}|\psi\rangle - c_B e^{i\tau}\bar{B}|\psi\rangle + c(|\psi\rangle - |\phi\rangle)\|^2 \geq 0 \, , \tag{12}$$

with $\bar{A} = A - \langle\psi|A|\psi\rangle$, $\bar{B} = B - \langle\psi|B|\psi\rangle$; c_A, c_B, c and τ being real numbers, and $|\phi\rangle$ being an arbitrary state. Calculating the modulus squared, we have

$$c_A^2 \Delta A^2 + c_B^2 \Delta B^2 \geq -\lambda c^2 - c_A c_B c\beta + c_A c_B \delta \, . \tag{13}$$

Here, ΔA^2 and ΔB^2 are the variances of A and B calculated on $|\psi\rangle$, respectively. $\lambda \equiv 2(1 - \mathrm{Re}[\langle\psi|\phi\rangle])$, $\beta \equiv 2\,\mathrm{Re}[\langle\psi|(-\bar{A}/c_B + e^{-i\tau}\bar{B}/c_A|\phi\rangle]$, and $\delta \equiv 2\,\mathrm{Re}[e^{i\tau}\langle\psi|\bar{A}\bar{B}|\psi\rangle]$. Choosing the value

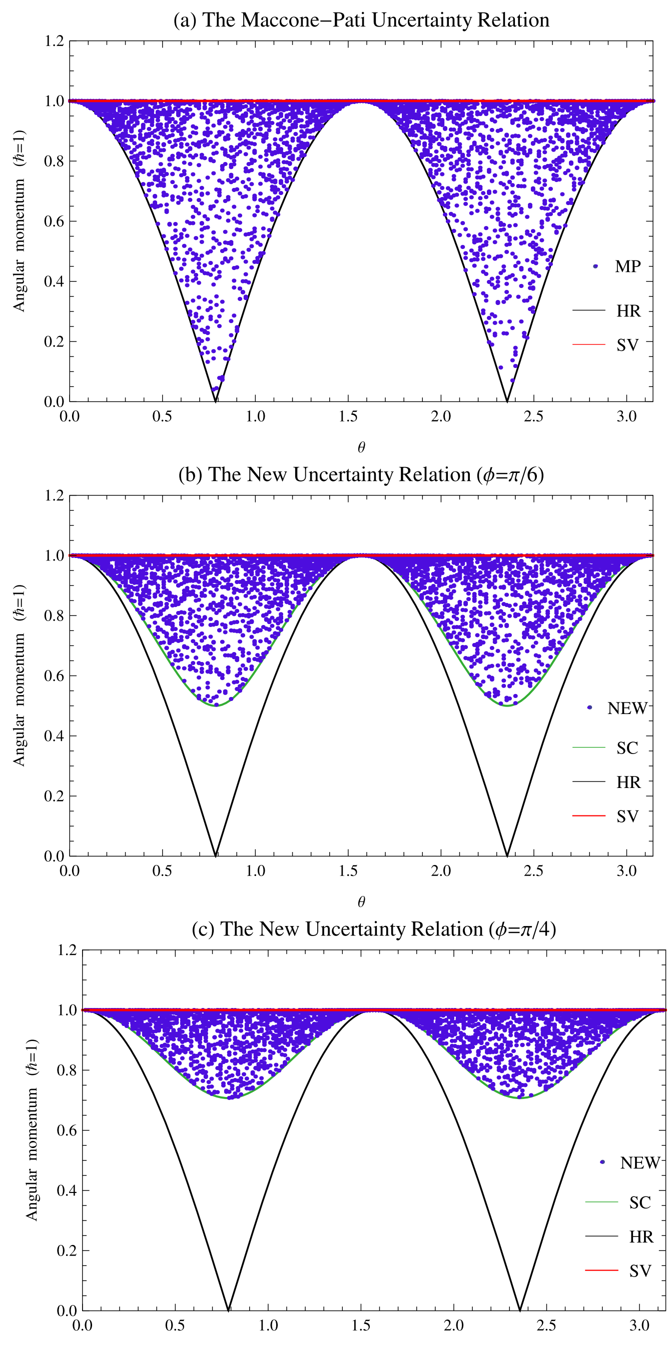

Fig. 1. Example of comparison between the Maccone–Pati uncertainty relation (MP) (3) and the new uncertainty relation (NEW) (9). Note that the new uncertainty relation (9) is stronger than the relation (3). We choose two components of the angular momentum $A = J_x$ and $B = J_y$ for a spin-1 particle, and a family of states parameterized by θ and ϕ as $|\psi\rangle = \cos\theta|1\rangle + \sin\theta e^{i\phi}|-1\rangle$, with $|\pm 1\rangle$ being eigenstates of J_z corresponding to the eigenvalues of ± 1. The upper red line denotes the sum of variances $\Delta J_x^2 + \Delta J_y^2$ (SV). The blue points exhibit domains of (3) in (a) and (9) in (b: $\phi = \pi/6$) and (c: $\phi = \pi/4$) with 20 randomly chosen states $|\psi^{\perp}\rangle$ for each of the 200 values of the phase θ. The green curve is the lower bound given by the Schrödinger uncertainty relation (SC) (10). The black curve is the lower bound set by the Heisenberg–Robertson uncertainty relation (HR) (1). The relation (3) gives the same results for any value of ϕ (a). If ϕ is not equal to 0 and π, the new uncertainty relation (9) always give nontrivial bound (b) and (c). When ϕ is equal to 0 or π, the relation (9) reduces to the relation (3) and have the same results as (a). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)