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Exact, rotational, infinite energy, blowup solutions to the 3-dimensional Euler equations

Manwai Yuen

Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong

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ABSTRACT

In this Letter, we construct a new class of blowup or global solutions with elementary functions to the 3-dimensional compressible or incompressible Euler and Navier–Stokes equations. And the corresponding blowup or global solutions for the incompressible Euler and Naiver–Stokes equations are also given. Our constructed solutions are similar to the famous Arnold–Beltrami–Childress (ABC) flow. The obtained solutions with infinite energy can exhibit the interesting behaviors locally. Furthermore, due to $\operatorname{div}\vec{u}=0$ for the solutions, the solutions also work for the 3-dimensional incompressible Euler and Navier–Stokes equations.

1. Introduction

The N-dimensional Euler equations can be formulated as the follows:

$$\begin{cases} \rho_t+\nabla\cdot(\rho\vec{u})=0 \\ \rho\big[\vec{u}_t+(\vec{u}\cdot\nabla)\vec{u}\big]+\nabla P(\rho)=0 \end{cases} \tag{1}$$

where $\vec{x}=(x_1,x_2,\dots,x_N)\in R^N$, $\rho=\rho(t,\vec{x})$ and $\vec{u}=\vec{u}(t,\vec{x})=(u_1,u_2,\dots,u_n)\in R^N$ are the density and the velocity respectively. The γ-law could be applied to the pressure function, i.e.

$$P(\rho)=K\rho^\gamma \tag{2}$$

with $K\geqslant 0$ and $\gamma\geqslant 1$. The Euler equations (1) can be rewritten by the scalar form,

$$\begin{cases} \dfrac{\partial\rho}{\partial t}+\displaystyle\sum_{k=1}^N u_k\frac{\partial\rho}{\partial x_k}+\rho\sum_{k=1}^N\frac{\partial u_k}{\partial x_k}=0, \\ \rho\left(\dfrac{\partial u_i}{\partial t}+\displaystyle\sum_{k=1}^N u_k\frac{\partial u_i}{\partial x_k}\right)+\dfrac{\partial P}{\partial x_i}=0, \quad \text{for } i=1,2,\dots,N. \end{cases} \tag{3}$$

The Euler equations (1) are the fundamental model in fluid mechanics [2] and [5].

Constructing exact solutions is a very important part in mathematical physics to understand the nonlinear behaviors of the system. For the pressureless fluids $K=0$, a class of exact blowup solutions were given in Yuen and Yeung's papers [14] and [11]:

$$\begin{cases} \rho=\dfrac{f(\frac{x_1+d_1}{a_1(t)},\frac{x_2+d_2}{a_2(t)},\dots,\frac{x_N+d_N}{a_N(t)})}{\prod_{i=1}^N a_i(t)} \\ u_i=\dfrac{\dot{a}_i(t)}{a_i(t)}(x_i+d_i) \quad \text{for } i=1,2,\dots,N \\ a_i(t)=a_{i0}+a_{i1}t \end{cases} \tag{4}$$

E-mail address: nevetsyuen@hotmail.com.

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with an arbitrary C^1 function $f \geqslant 0$, and constants d_i, $a_{i0} > 0$ and a_{i1}.

In particular, for $a_2 < 0$, the solutions blowup in the finite time $T = -a_1/a_2$. There are other analytical solutions for the compressible Euler or Navier–Stokes equations in [9,8,6,12–14,11,15] and [16]. In addition, there are two papers to investigate for a special structural form which generalizes the exact solutions for Burgers' vortices in [7] and [3].

In this Letter, we manipulate the elementary functions to construct some exact rotational solutions for the 3-dimensional compressible Euler equations (1):

Theorem 1. *For the* 3*-dimensional compressible Euler equations* (1) *with* $K > 0$, *there exists a class of rotational solutions*:

For $\gamma > 1$:

$$\begin{cases} \rho = \max\left\{\dfrac{\gamma-1}{K\gamma}\left[C^2\left[x^2+y^2+z^2-(xy+yz+xz)\right]-\dot{a}(t)(x+y+z)+b(t)\right], 0\right\}^{\frac{1}{\gamma-1}} \\ u_1 = a(t) + C(y-z) \\ u_2 = a(t) + C(-x+z) \\ u_3 = a(t) + C(x-y) \end{cases} \tag{5}$$

where

$$a(t) = c_0 + c_1 t \tag{6}$$

and

$$b(t) = 3c_0c_1t + \frac{3}{2}c_1^2t^2 + c_2 \tag{7}$$

with C, c_0, c_1 *and* c_2 *arbitrary constants*;

For $\gamma = 1$:

$$\begin{cases} \rho = e^{\frac{1}{K}[C^2[x^2+y^2+z^2-(xy+yz+xz)]-\dot{a}(t)(x+y+z)+b(t)]} \\ u_1 = a(t) + C(y-z) \\ u_2 = a(t) + C(-x+z) \\ u_3 = a(t) + C(x-y). \end{cases} \tag{8}$$

The solutions (5) *and* (8) *globally exist.*

Remark 2. In 1965, Arnold first introduced the famous Arnold–Beltrami–Childress (ABC) flow

$$\begin{cases} u_1 = A\sin z + C\cos y \\ u_2 = B\sin x + A\cos z \\ u_3 = C\sin y + B\cos x \end{cases} \tag{9}$$

with constants A, B, C and a suitable pressure function P only for the incompressible Euler equations in [1]. We observe that our solutions (5) and (8) are similar to the ABC flow.

Remark 3. The solutions with infinite energy can exhibit the interesting behaviors locally. The exact solutions with infinite energy of the systems may be regionally applicable to understand the great complexity that exists in turbulent phenomena.

Remark 4. We notice that the rational functional form with $a(t) = 0$ and $C = 1$ in solutions (5) and (8) for the velocity $\vec{u}$ comes from Senba and Suzuki's book [10]. The velocities $\vec{u}$ in solutions (5) and (8) are not spherically symmetric.

Remark 5. The exact rotational solutions (5) and (8) could be good examples for testing numerical methods for fluid dynamics.

2. Compressible rotational fluids

The main technique of this article is just to use the primary assumption about the velocities $\vec{u}$:

$$\begin{cases} u_1 = a(t) + C(y-z) \\ u_2 = a(t) + C(-x+z) \\ u_3 = a(t) + C(x-y) \end{cases} \tag{10}$$

to substitute the governing equations (1) to construct the density function ρ to balance the system. Therefore, the proof is simple to be checked by direct computation:

Proof of Theorem 1. For the vacuum solutions $\rho = 0$, it is the trivial solutions for the system (1).

For $\gamma > 1$, with non-vacuum solutions, we have the first momentum equation $(1)_{2,1}$:

$$\frac{\partial}{\partial t}u_1 + u_1\frac{\partial}{\partial x}u_1 + u_2\frac{\partial}{\partial y}u_1 + u_3\frac{\partial}{\partial z}u_1 + K\gamma\rho^{\gamma-2}\frac{\partial}{\partial x}\rho \tag{11}$$

$$= \frac{\partial}{\partial t}u_1 + u_2\frac{\partial}{\partial y}u_1 + u_3\frac{\partial}{\partial z}u_1 + \frac{K\gamma}{\gamma-1}\frac{\partial}{\partial x}\rho^{\gamma-1} \tag{12}$$