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Innovation diffusion equations on correlated scale-free networks

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ABSTRACT

We introduce a heterogeneous network structure into the Bass diffusion model, in order to study the diffusion times of innovation or information in networks with a scale-free structure, typical of regions where diffusion is sensitive to geographic and logistic influences (like for instance Alpine regions). We consider both the diffusion peak times of the total population and of the link classes. In the familiar trickle-down processes the adoption curve of the hubs is found to anticipate the total adoption in a predictable way. In a major departure from the standard model, we model a trickle-up process by introducing heterogeneous publicity coefficients (which can also be negative for the hubs, thus turning them into stiflers) and a stochastic term which represents the erratic generation of innovation at the periphery of the network. The results confirm the robustness of the Bass model and expand considerably its range of applicability.

1. Introduction

Like the spreading of diseases, the diffusion of innovation and information in a population can be analyzed through several mathematical models. "Compartimental" models divide the population into groups of individuals who are in certain states: sound/infected, ignorant/informed etc. The evolution in time of the populations of the various compartments is described by differential equations. In the basic version of the models, the population of each compartment is seen as homogeneous. A possible improvement consists in the introduction of a network structure; this has been done, among others, by Boguna et al. [1,2] in the statistical networks formalism (for epidemics) and by Moreno et al. [3,4] in the stochastic network formalism (for information). Another possible approach is to assign a network whose nodes are regarded as two-state systems, and to write master equations giving the probability that a node passes from one state to the other, as a consequence of its internal dynamics and of the state of its neighbours, like in cellular automata models. With a method of this kind, Gleeson [5] has found exact solutions of the Bass diffusion model, which is based, in the homogeneous version, upon a linear publicity term and a quadratic imitation term.

The Bass model has been extensively employed in marketing and social sciences [6,7], resulting in large empirical databases of its coefficients, measured in different situations. It has also been employed for agent-based simulations of social networks modelled on real networks [8] in order to check the effect of the so-called "influencers". In this work we apply to the Bass model the statistical network description which has been successfully employed for epidemics. Our main aim is to follow diffusion in time, from its beginning to its end, in particular in order to see when the diffusion peak occurs, in dependence on the exponent γ of scale-free networks and on their assortative or disassortative correlations. We also study the temporal evolution in each link class, namely we measure the diffusion times for individuals who are more or less connected, and give quantitative estimates of the anticipation of diffusion in the most connected classes.

Our aim is therefore different from that of epidemic models, where one is mostly interested into the initial phases of the epidemic, into the epidemic threshold and into its dependence on the model parameters. Our aim is also different from that of the Gleeson method, which allows to find phases in parameters space, and from that of Moreno et al., which measures the reliability of diffusion and the load on the network, as compared to deterministic diffusion.

In recent work using this approach [9] we found that the hubs can serve as "monitors" for the adoption of others and may allow to estimate the parameters of a diffusion process at its beginning, or anyway long before the global peak, in the case when no empirical parameters are available. A technical difficulty encountered is the explicit construction of the matrices $P(h|k)$ appearing in the network Bass equations. ($P(h|k)$ gives the conditional probability

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that a node with k links is connected to one with h links.) It is not sufficient for our purposes to consider the nearest-neighbour average function $\bar{k}_{nn} = \sum_h hP(h|k)$, as done in [1].

In the uncorrelated case it is straightforward to write the matrices $P(h|k)$, but in the assortative and disassortative case this has never been done before, or not for large and arbitrary dimension. We have devised and given detailed analytical proofs for a method which works especially well in the assortative case, the one actually relevant for most social networks. This allowed us in [9] to make quantitative comparisons with the uncorrelated case (and also with the disassortative case, when applicable).

A further improvement made possible by the network structure is to make the publicity term p in the Bass equation heterogeneous. The most straightforward way to extend the Bass model in this direction is to enhance its "trickle-down" character, by defining p_i coefficients which are larger for the hubs, while still giving the some total expenditure for publicity.

Sometimes, however, innovation proceeds from the periphery of a system, where it is generated and grows in small niches, until it becomes the "disruptive" innovation described by Christensen [11]. In that case, the innovation is not launched through a marketing campaign, but starts as a stochastic and erratic process; then diffusion can accelerate or almost come to a halt several times at random, before it reaches some points in the network where it can propagate more vigorously. Empirical evidence [12] shows that this is often the case for hierarchical inter-firm structures like those common in Alpine regions for geographical and logistic reasons [13]. This "trickle-up" process is the main subject of this work and can be modelled through p_i coefficients which are larger for the less connected nodes of the network.

One can also take into account the possibility that some of the hubs are "conservative" and act as bottlenecks for diffusion, instead of facilitating it. This can be modelled through negative publicity coefficients, leading to a major departure from the familiar Bass model, where one has to careful adapt the equations in such a way to allow temporary negative values of the adoption rate f, but avoid negative values of the cumulative adoption F, which would be meaningless.

Finally, since our model allows to follow the detailed time evolution of the system through the numerical solution of the N differential equations, we can add a stochastic term and study the resulting Langevin equations, in particular in the case of "trickle-up" innovation [10].

The outline of the article is the following. In Sect. 2 we recall the network Bass equations introduced in [9] and the main features of its numerical solutions. In Sect. 3 we introduce the new trickle-up equations and give some numerical solutions. In Sect. 4 we introduce the stochastic trickle-up equations. Sect. 5 contains a general discussion of the possible applications of the model and our conclusions.

2. Network structure and trickle-down

The network Bass equation introduced in [9] is a system of nonlinear first order differential equations, in which the imitation term of the Bass model has been split over N connectivity classes:

$$\frac{dG_i(t)}{dt} = [1 - G_i(t)] \left[p + iq \sum_{h=1}^{N} P(h|i) G_h(t) \right] \qquad i = 1, \dots, N. \tag{1}$$

$G_i(t) = F_i(t)/P(i)$ is the fraction of potential adopters with i links that at the time t have actually adopted the innovation. The matrices $P(i) = c/i^\gamma$ (link density) and $P(h|i)$ (link correlations) must obey the Network Closure Condition (NCC; see [14]). Further conditions are normalization $\sum_{k=1}^{N} P(k|i) = 1$, $\sum_{k=1}^{N} P(k) = 1$ and positivity $P(i|k) \geq 0$, $P(i) \geq 0$.

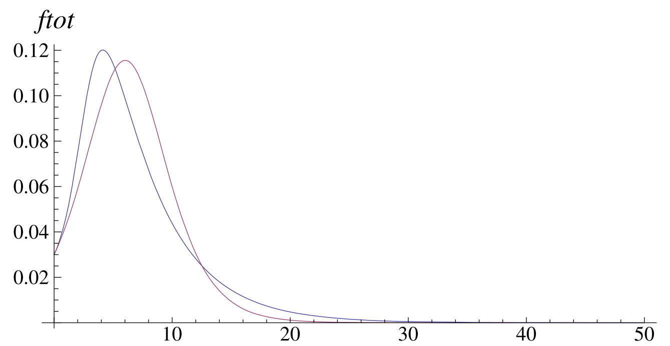

Fig. 1. Total adoptions f_{tot} in time (blue curve), for an uncorrelated scale-free network with density proportional to $1/i$ ($\gamma = 1$), with $N = 15$. The violet curve is the simple Bass function with the same q and p ($q = 0.4$, $p = 0.03$). The function f_{tot} peaks approx. at $t = 4.4$, while the simple Bass function peaks approx. at $t = 6.1$. (For interpretation of the references to colour in this figure, the reader is referred to the web version of this article.)

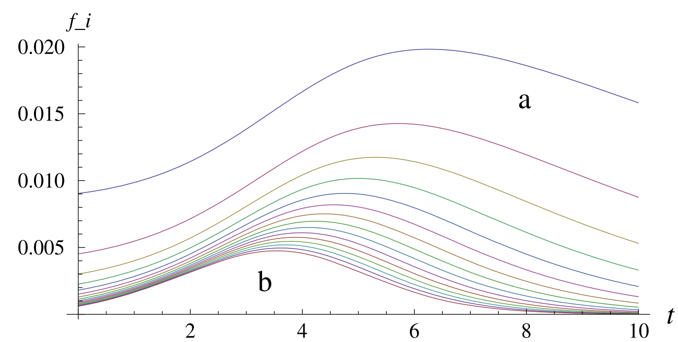

Fig. 2. Partial adoptions in time in the different link classes for the same parameters as in Fig. 1. The function (a) which has the largest value at $t = 0$ is f_1, representing the adoptions of individuals with only 1 link. The function f_{15}, which represents the adoptions of the most connected individuals (b), peaks approx. at $t = 3.5$.

Table 1
Dependence on the scale-free exponent γ of the total diffusion peak time T and of the partial diffusion peak time T_{50} (for individuals with 50 links) in a network with largest degree $N = 100$, in the uncorrelated and assortative case. The peak time T_{50} has been chosen as indicator of the anticipated diffusion in the hubs, instead of T_{100}, because in these networks the hubs with 100 links are a very small fraction of the total population, and thus are not significant for statistical monitoring. Nevertheless, the anticipation effect is clearly very strong, especially for the assortative networks: for $\gamma = 2$, T_{50} is approx. 20% of T, and for $\gamma = 3$ it is approx. 10% of T. Note that for uncorrelated networks the total diffusion time is smaller, but the anticipation effect is weaker. The dependence $T(\gamma)$ in the case of variable p coefficients $p_k = c_1 k^\gamma p$ is also shown.

	$\gamma = 2.0$	$\gamma = 2.25$	$\gamma = 2.5$	$\gamma = 2.75$	$\gamma = 3.0$
T-Assort.	2.5	4.0	4.8	5.1	5.4
T_{50}-Assort.	0.80	0.64	0.51	0.49	0.49
T-Uncorr.	2.4	2.8	3.5	4.2	4.7
T_{50}-Uncorr.	1.4	1.3	1.3	1.3	1.3
T-Uncorr., var. p	1.6	2.5	3.6	4.8	5.9

2.1. Summary of trickle-down results

A summary of the numerical results obtained in [9] for the trickle-down case is given in Figs. 1, 2 and Table 1. Table 2 summarizes results for a wider range of the exponent γ and includes for comparison results for the trickle-up case, to be discussed in the next section.