Physics Letters A 380 (2016) 2480–2484

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Semiclassical and quantum description of an ideal Bose gas in a uniform gravitational field

Rajat K. Bhaduri [a], Wytse van Dijk [a,b,*]

[a] Department of Physics and Astronomy, McMaster University, Hamilton, ON, L8S 4M1 Canada
[b] Department of Physics, Redeemer University College, Ancaster, ON, L9K 1J4 Canada

ARTICLE INFO

Article history:
Received 15 April 2016
Received in revised form 26 May 2016
Accepted 27 May 2016
Available online 1 June 2016
Communicated by C.R. Doering

Keywords:
BEC
Uniform gravity
Isothermal compressibility

ABSTRACT

We consider an ideal Bose gas contained in a cylinder in three spatial dimensions, subjected to a uniform gravitational field. It has been claimed by some authors that there is discrepancy between the semiclassical and quantum calculations in the thermal properties of such a system. To check this claim, we calculate the heat capacity and isothermal compressibility of this system semiclassically as well as from the quantum spectrum of the density of states. The quantum calculation is done for a finite number of particles. We find good agreement between the two calculations when the number of particles are taken to be large. We also find that this system has the same thermal properties as an ideal five dimensional Bose gas.

1. Introduction

There is a body of literature on an ideal Bose gas in a uniform gravitational field [1–5]. The gas may be contained in an external potential, or a large box, and subjected to a uniform gravitational potential. Using the semiclassical approximation, its thermal properties have been calculated analytically in the grand canonical formalism [6,7]. Our motivation for studying this simple system is to check how closely does the semiclassical approximation follow the results of the quantum calculation. The authors of Ref. [4] claim that, contrary to previous wisdom, in three dimensions an ideal Bose gas in a uniform gravitational field does not undergo BEC at a finite temperature. They attribute that to the replacement of the discrete quantum energy spectrum with a smooth density of states. In the semiclassical approximation, one replaces the discrete density of states by a smooth one, while treating the ground state exactly. In the quantum calculation, on the other hand, the exact discrete energy levels of the system are calculated to compute the grand canonical ensemble (GCE) and the resulting thermal properties. In realistic statistical mechanics problems, one generally follows the semiclassical route. For the system at hand, the quantum calculation is done for a finite number of particles. We find that as the number of particles is increased to larger and larger values, the quantum and semiclassical results become very close, even across the BEC critical temperature.

In this paper, we pay special attention to the calculation of the isothermal compressibility of the Bose gas. Recent experimental work on the isothermal compressibility across the Bose–Einstein condensation has been reported in Ref. [8] for a harmonically trapped gas. The authors suggest that the isothermal compressibility around the critical pressure reveals a second-order nature of the phase transition. On the other hand predictions based on a number of different mean-field approximations [9] do not lead to second-order phase transitions, and the isothermal compressibility does not diverge at criticality. In contrast to these authors, we discuss noninteracting systems only. It is well documented that in GCE, the isothermal compressibility diverges at the critical temperature T_c in the absence of interparticle interactions [10]. It is also known that even a weak interparticle interaction removes this divergence [11]. In the present problem, however, gravitation is introduced as a one-body ramp potential, and it is not clear at the outset how it will affect the compressibility.

We find that the semiclassical calculation in three dimensions of the ideal Bose gas with uniform gravity is equivalent to the analysis of a five-dimensional ideal Bose gas without gravity. We use this novel approach to obtain results for the specific heat and isothermal compressibility. The resulting compressibility is divergence-free and continuous across T_c. In the case of heat capacity, in the absence of the gravitational field, there is a discontinuity in its slope at T_c. Introducing gravitation, or, alternately five spatial dimensions, this discontinuity is in the heat capacity itself.

* Corresponding author at: Department of Physics and Astronomy, McMaster University, Hamilton, ON, L8S 4M1 Canada.
E-mail address: vandijk@physics.mcmaster.ca (W. van Dijk).

http://dx.doi.org/10.1016/j.physleta.2016.05.055
0375-9601/

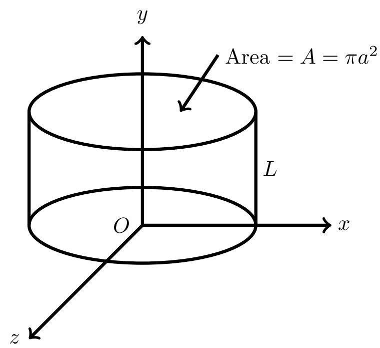

Fig. 1. The Bose gas is confined to a cylindrical box with the downward gravitational force parallel to the y axis.

The calculations were performed by taking a cylindrical container, as shown in Fig. 1. In the zx plane, we take a circular disc, which is the bottom of the cylinder at $y = 0$. The atoms in the Bose gas are not allowed to take negative values of y. The gravitational field is along the y direction, and the potential is a ramp along the positive y axis.

The plan of the paper is as follows. In Sec. 2, the semiclassical calculation is done using the phase space approach. It is established that one can describe the system under consideration in five spatial dimensions, but without the gravitation. The grand potential is calculated and the critical temperature T_c is obtained. In Sec. 3, we give the results for isothermal compressibility and the heat capacity. In Sec. 4 a quantum calculation is done to show that BEC takes place and the results agree with the semiclassical calculation.

2. Three-dimensional gas in a uniform gravitational potential

In this section, we show that an ideal Bose gas in three spatial dimensions, subjected to a uniform gravitational potential, may be looked upon as an ideal five dimensional gravity-free gas. We then use the semiclassical method to calculate the critical temperature of BEC.

Using the geometry of Fig. 1, the single particle energy is given by

$$\epsilon(p, y) = \frac{p^2}{2m} + mgy, \tag{1}$$

where m is the mass of each boson, and g is the gravitational acceleration on the earth's surface, and $p^2 = (p_x^2 + p_y^2 + p_z^2)$. The grand potential is given by ($k_B = 1$)

$$\Omega_b = T \sum_n \ln(1 - z\exp(-\epsilon_n\beta)) = -T \sum_{l=1}^{\infty} \frac{(z)^l}{l} Z_1(l\beta), \tag{2}$$

where $\beta = 1/T$, the fugacity $z = \exp(\beta\mu)$, and $Z_1(l\beta)$ is the one-body partition function in the variable $l\beta$. In the semiclassical approximation, $Z_1(\beta)$ in the variable β, is given by

$$Z_1(\beta) = \frac{1}{h^3} \int d^3p\, e^{-\beta p^2/2m} \int d^2r \int_0^L dy\, e^{-\beta mgy}. \tag{3}$$

The two-dimensional spatial integral gives the area A of the disc, yielding

$$Z_1(\beta) = \frac{A}{\lambda_T^3} \frac{(1 - \exp(-\beta mgL))}{\beta mg}. \tag{4}$$

Note that as $g \to 0$, we recover the correct $Z_1(\beta) = \frac{V}{\lambda_T^3}$, where $V = AL$ is the three-dimensional spatial volume. The thermal wavelength λ_T (obtained from the p integration) is given by

$$\lambda_T = \sqrt{\frac{2\pi\hbar^2}{mT}}. \tag{5}$$

For our present problem with *nonzero* g and low temperatures, we impose the condition that $k_B T \ll mgL$, i.e. $\beta mgL \gg 1$. Under this condition, Eq. (4) reduces to

$$Z_1(\beta) = \frac{A}{\lambda_T^3} \frac{1}{\beta mg}. \tag{6}$$

Equation (6) could be rewritten as an ideal five-dimensional partition function (without gravity)

$$\tilde{Z}_1(\beta) = \frac{V_5}{\lambda_T^5} \tag{7}$$

where

$$V_5 = \frac{2\pi\hbar^2 A}{m^2 g} \tag{8}$$

has the dimension of (length)^5. We write $V_5 = (A \cdot V_3)$, where V_3 is a hypothetical 3-volume. Taking m to be that of a Rb^{87} atom, we find V_3 to be exceedingly small, of the order of 10^{-18} cubic meter. This V_3 is not to be confused with the large three-dimensional volume $V = AL$ in which the atoms are confined. In the following, we shall calculate the thermal properties of this noninteracting gas of bosons in 5-spatial dimensions.

Substituting for $\tilde{Z}_1(\beta)$ from Eq. (7) in Eq. (2), we see that the grand potential may be written as

$$\Omega_b = -T \frac{V_5}{\lambda_T^5} \sum_{l=1}^{\infty} \tilde{b}_l z^l = -T \frac{V_5}{\lambda_T^5} g_{7/2}(z), \tag{9}$$

where $\tilde{b}_l = 1/l^{7/2}$ are the statistical "cluster integrals". In standard notation, $\sum_{l=1}^{\infty} z^l/l^{7/2} = g_{7/2}(z)$.

In the gas phase,

$$\bar{n}_5 = \frac{\bar{N}}{V_5} = -\frac{\partial\Omega_b}{\partial\mu} = \frac{1}{\lambda_T^5} \sum_{l=1} \tilde{b}_l z^l = \frac{1}{\lambda_T^5} g_{5/2}(z). \tag{10}$$

One puts in the constraint that $\bar{N} = N$, and this makes z a function of T. The sum on the RHS converges at $z = 1$, so the above relation is valid only for $T \geq T_c$. For lower temperatures, the ground state starts having macroscopic occupancies. The critical temperature is given by

$$(\bar{n}_5 \lambda_{T_c}^5) = \zeta(5/2), \tag{11}$$

where λ_T is at $T = T_c$, and $\zeta(5/2)$ is the Riemann zeta function. It is straightforward to deduce from Eq. (11) that the critical temperature is given by

$$T_c = \left(\frac{\bar{N}\hbar^3 g(2\pi)^{3/2}}{\zeta(5/2) A\sqrt{m}} \right)^{2/5}. \tag{12}$$

This agrees with the expression for T_c as given by Du et al. [7], that was obtained by the standard procedure in three spatial dimensions in the presence of the uniform gravitational field. Furthermore it follows that

$$\frac{N_{\epsilon=0}}{N} = 1 - \left(\frac{T}{T_c}\right)^{5/2} \quad \text{when} \quad T < T_c, \tag{13}$$