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# Full characterization of modular values for finite-dimensional systems

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ABSTRACT

Kedem and Vaidman obtained a relationship between the spin-operator modular value and its weak value for specific coupling strengths [14]. Here we give a general expression for the modular value in the $n$-dimensional Hilbert space using the weak values up to $(n-1)$th order of an arbitrary observable for any coupling strength, assuming non-degenerated eigenvalues. For two-dimensional case, it shows a linear relationship between the weak value and the modular value. We also relate the modular value of the sum of observables to the weak value of their product.

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## 1. Introduction

The "weak value" is a groundbreaking concept that was discovered by Aharonov, Albert, and Vaidman [1]. The weak value of an observable $\hat{A}$ is defined as the expectation value of $\hat{A}$ via weak measurements performed between the pre-selection of an initial state and the post-selection of a final state. The weak value of $\hat{A}$ is given by $\langle\hat{A}\rangle_{\rm w} = \langle\phi|\hat{A}|\psi\rangle/\langle\phi|\psi\rangle$. Contrary to the usual expectation values, a weak value might even lie far outside the range of eigenvalues of the observable $\hat{A}$ and can even be a complex number. The properties of the weak value have been extensively studied from theoretical and experimental points of view in recent years. Particularly, the cases of nonlocal observables are interesting [2–4], including EPR paradox [5–7], Hardy's paradox [8–11], and Cheshire Cat experiment [12,13].

The most studies on weak values focus on a continuous-variable pointer of the measuring device, which distributes in Gaussian distribution, and assume that the interaction between the system and the meter is weak. Y. Kedem and L. Vaidman, however, recently considered the cases where the meter is a qubit, and also the interaction between the system and meter qubit is arbitrarily strong [14]. The system, which does not have to be a qubit, is conditioned by an initial state vector $|\psi\rangle$ and a final state vector $|\phi\rangle$ [15], and the state of the meter qubit is initially prepared to be $\gamma|0\rangle + \bar{\gamma}|1\rangle$ ($\gamma$ and $\bar{\gamma}$ are real numbers satisfying $\gamma^2 + \bar{\gamma}^2 = 1$), with $\bar{\gamma} \ll 1$.

The interaction Hamiltonian is written as

$$\hat{H} = \hbar g(t)\hat{A}\otimes\hat{\Pi}\,, \quad \text{with} \int g(t){\rm d}t = g\,, \tag{1}$$

where, $\hat{\Pi} \equiv |1\rangle\langle 1|$ denotes the projection operator onto state $|1\rangle$ of the meter qubit, $\hat{A}$ represents the Hermitian operator corresponding to the observable of the quantum system, and the coupling $g(t)$ generally is a time varying function, the resulting coupling constant $g$ can be arbitrarily large.

The final state of the meter qubit is calculated as follows

$$\begin{aligned}
&\langle\phi|e^{-ig\hat{A}\otimes|1\rangle\langle 1|}|\psi\rangle(\gamma|0\rangle + \bar{\gamma}|1\rangle)\\
&= \langle\phi|\begin{pmatrix}\hat{I} & 0\\ 0 & e^{-ig\hat{A}}\end{pmatrix}|\psi\rangle(\gamma|0\rangle + \bar{\gamma}|1\rangle)\\
&= \begin{pmatrix}\langle\phi|\psi\rangle & 0\\ 0 & \langle\phi|e^{-ig\hat{A}}|\psi\rangle\end{pmatrix}(\gamma|0\rangle + \bar{\gamma}|1\rangle)\\
&= \langle\phi|\psi\rangle\left[\gamma|0\rangle + \bar{\gamma}\,\frac{\langle\phi|e^{-ig\hat{A}}|\psi\rangle}{\langle\phi|\psi\rangle}|1\rangle\right],
\end{aligned} \tag{2}$$

where, we have used the bases $|0\rangle = \binom{1}{0}$ and $|1\rangle = \binom{0}{1}$. The complex number $\frac{\langle\phi|e^{-ig\hat{A}}|\psi\rangle}{\langle\phi|\psi\rangle}$ was named the "modular value" of operator $\hat{A}$ [14], which is written as $(\hat{A})_{\rm m}$. Therefore,

$$(\hat{A})_{\rm m} \equiv \frac{\langle\phi|e^{-ig\hat{A}}|\psi\rangle}{\langle\phi|\psi\rangle}\,. \tag{3}$$

The modular value has the same amplification factor $1/\langle\phi|\psi\rangle$ as the weak value. Nevertheless, in some cases, the modular value

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can be related to the weak value. Let us give an example of spin operators $\hat{\sigma}_x, \hat{\sigma}_y$ and $\hat{\sigma}_z$ with $g = -\pi/2$. We have [14]:

$$(\hat{\sigma})_{\mathrm{m}} \equiv \frac{\langle\phi|e^{i\frac{\pi}{2}\hat{\sigma}}|\psi\rangle}{\langle\phi|\psi\rangle} = i\langle\hat{\sigma}\rangle_{\mathrm{w}} \quad (\hat{\sigma} = \hat{\sigma}_x, \hat{\sigma}_y \text{ or } \hat{\sigma}_z) . \tag{4}$$

Therefore, the modular value of a spin component is directly related to its weak value in this specific case, i.e., $\hat{A} = \hat{\sigma}$ and $g = -\pi/2$ [14]. From the point of view of measurement, a modular value is easily obtained because one can simply perform the tomography using the binary outcomes of the meter qubit. On top of it, a modular value can be measured more efficiently than a weak value because the measurement coupling constant $g$ can be made large.

However, the fundamentals of modular values are not fully understood yet. For example, a general expression for the relation between the modular value and the weak value of an observable is still missing. Moreover, the cases that the system's Hilbert space is two-dimensional, i.e. spin-1/2 or qubit systems, have been studied extensively and there are many implications in quantum mechanics and quantum computation [16–19]. Recently, Lorenzo has also studied quantum measurements in three-dimensional Hilbert space, such as a spin-1 system (or a qutrit) [20]. In the usual quantum information processing, a qubit plays the role of unit of quantum information, whereas, in quantum cryptography, there are reports claim that qutrit systems are more secure than the qubit systems [21,22]. It is thus desirable to investigate modular values and weak values not only in the two-dimensional systems but also in higher dimensional systems.

In this Letter, we obtain more generalized expressions for the relationship between the modular values and the weak values by using the Lagrange interpolation. For example, in two-dimensional cases, we generalize Eq. (4) to the one for an arbitrary coupling constant. This expression, of course, reproduces the previous report [14] for specific coupling strengths. We show that, in general, not only a single weak value but also a joint weak value of a two-dimensional system can be expressed by the modular values. For three-dimensional systems, we show the modular value of an observable can be expressed by the weak value of the observable and the square of the observable. We then generalize the theory to $n$-dimensional systems, and show the modular value can be expressed in term of weak values up to $(n-1)$th order.

Our general method allows us to deal with both local and nonlocal measurements, and explains the anomalous results of some intriguing experiments, as are described in Sec. 3. To illustrate, we consider the nonlocal joint weak values for two-dimensional systems and show that the joint weak values can be obtained by measuring proper modular values. Additionally, a simple quantum circuit that simulates the measurement of modular value has been examined, where a controlled rotation gate plays the role of modular value interaction $\hat{U}$ and the half rotation angle stands for the coupling constant $g$.

The rest of this Letter is organized as follows. General expressions to relate the weak values and the modular values are shown in Sec. 2. In Sec. 3 we examine the nonlocal joint weak value for two-dimensional Hilbert space with the aid of modular values. Then, we give some interesting examples such as EPR paradox, Hardy's paradox, and Cheshire Cat experiments. We finally consider a controlled-$R_z(\theta)$ gate in Sec. 4, where the meter qubit controls the system qubit, to realize the measurement of the modular value of $\hat{\sigma}_z$ of the system qubit. The Letter concludes with remarks in Sec. 5.

## 2. General expressions to relate weak and modular values

Our first main result is that, when the dimension of the Hilbert space $n$ is two, the weak value of an arbitrary observable can be calculated from its modular value, and vice versa, for any quantity of the coupling constant. When the dimension of Hilbert space is larger than two, we show that the modular value can be expressed in term of weak values up to $(n-1)$th order.

Let us first start from the case of $n(\geq 2)$dimensional Hilbert space. We also assume that an arbitrary observable $\hat{A}$ has $n$ different eigenvalues $\lambda_k$ $(k = 1, 2, ..., n)$, which are known. Note that throughout this letter, we consider the case of non-degeneracy of eigenvalues. We now introduce the Lagrange interpolation of the matrix form [23]:

$$e^{-igA} = \sum_{k=1}^{n} e^{-ig\lambda_k} \prod_{\ell=1,\ell\neq k}^{n} \frac{A-\lambda_\ell I}{\lambda_k - \lambda_\ell} , \tag{5}$$

where $A$ is the matrix expression of $\hat{A}$, and $I$ is the unit matrix. Taking the eigenvectors of $\hat{A}$ as the bases for the matrix expression, Eq. (5) immediately leads to the interpolation of operator form as

$$e^{-ig\hat{A}} = \sum_{k=1}^{n} e^{-ig\lambda_k} \prod_{\ell=1,\ell\neq k}^{n} \frac{\hat{A}-\lambda_\ell \hat{I}}{\lambda_k - \lambda_\ell} . \tag{6}$$

### 2.1. Two-dimensional Hilbert space

Particularly for $n = 2$, such as spin-1/2 particles, an arbitrary observable $\hat{A}$ has two distinguishable eigenvalues $\lambda_1$ and $\lambda_2$, and then Eq. (6) explicitly becomes

$$\begin{aligned} e^{-ig\hat{A}} &= e^{-ig\lambda_1}\frac{\hat{A}-\lambda_2\hat{I}}{\lambda_1-\lambda_2} + e^{-ig\lambda_2}\frac{\hat{A}-\lambda_1\hat{I}}{\lambda_2-\lambda_1} \\ &= \frac{\lambda_1 e^{-ig\lambda_2} - \lambda_2 e^{-ig\lambda_1}}{\lambda_1-\lambda_2}\hat{I} + \frac{e^{-ig\lambda_1}-e^{-ig\lambda_2}}{\lambda_1-\lambda_2}\hat{A} \\ &= \Lambda\hat{I} + \Lambda'\hat{A} , \end{aligned} \tag{7}$$

where

$$\Lambda = \frac{\lambda_1 e^{-ig\lambda_2} - \lambda_2 e^{-ig\lambda_1}}{\lambda_1-\lambda_2}$$

and $\Lambda' = \dfrac{e^{-ig\lambda_1}-e^{-ig\lambda_2}}{\lambda_1-\lambda_2}$

are complex numbers.

Applying pre- and post-selected states, $|\psi\rangle$ and $\langle\phi|$, from the right side and the left side of Eq. (7), respectively, we obtain the modular value of $\hat{A}$ in relation to its weak value as

$$(\hat{A})_{\mathrm{m}} = \Lambda + \Lambda'\langle\hat{A}\rangle_{\mathrm{w}} . \tag{8}$$

Inversely solving this, it is straightforward to express the weak value of $\hat{A}$ by its modular value as

$$\langle\hat{A}\rangle_{\mathrm{w}} = [(\hat{A})_{\mathrm{m}} - \Lambda]/\Lambda' . \tag{9}$$

As the first illustration, let us check whether this reproduces the relation between the weak value and the modular value of a spin-$\frac{1}{2}$ operator $\hat{\sigma}$ $(= \hat{\sigma}_x, \hat{\sigma}_y,$ or $\hat{\sigma}_z)$ in the case of $g = -\pi/2$. The spin operator has two eigenvalues: $\lambda_1 = 1$ for $|\uparrow\rangle$ and $\lambda_2 = -1$ for $|\downarrow\rangle$. Then, the modular value of $\hat{\sigma}$ is immediately given by Eq. (8) as

$$\begin{aligned} (\hat{\sigma})_{\mathrm{m}} &= \frac{e^{i\frac{\pi}{2}} - e^{-i\frac{\pi}{2}}}{2}\langle\hat{\sigma}\rangle_{\mathrm{w}} + \frac{e^{i\frac{\pi}{2}} + e^{-i\frac{\pi}{2}}}{2} \\ &= i\langle\hat{\sigma}\rangle_{\mathrm{w}} . \end{aligned} \tag{10}$$

This is exactly the result obtained by [14] shown in Eq. (4) of the present Letter.

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