Physics Letters A 380 (2016) 2142–2149

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

The Lyapunov dimension and its estimation via the Leonov method

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ARTICLE INFO

Article history:
Received 26 February 2016
Received in revised form 20 April 2016
Accepted 21 April 2016
Available online 25 April 2016
Communicated by C.R. Doering

Keywords:
Attractors of dynamical systems
Hausdorff dimension
Lyapunov dimension Kaplan–Yorke formula
Finite-time Lyapunov exponents
Invariance with respect to diffeomorphisms
Leonov method

ABSTRACT

Along with widely used numerical methods for estimating and computing the Lyapunov dimension there is an effective analytical approach, proposed by G.A. Leonov in 1991. The Leonov method is based on the direct Lyapunov method with special Lyapunov-like functions. The advantage of the method is that it allows one to estimate the Lyapunov dimension of invariant sets without localization of the set in the phase space and, in many cases, to get effectively an exact Lyapunov dimension formula. In this work the invariance of the Lyapunov dimension with respect to diffeomorphisms and its connection with the Leonov method are discussed. For discrete-time dynamical systems an analog of Leonov method is suggested. In a simple but rigorous way, here it is presented the connection between the Leonov method and the key related works: Kaplan and Yorke (the concept of the Lyapunov dimension, 1979), Douady and Oesterlé (upper bounds of the Hausdorff dimension via the Lyapunov dimension of maps, 1980), Constantin, Eden, Foiaş, and Temam (upper bounds of the Hausdorff dimension via the Lyapunov exponents and Lyapunov dimension of dynamical systems, 1985–90), and the numerical calculation of the Lyapunov exponents and dimension.

1. Introduction

The concept of the Lyapunov dimension was suggested in the seminal paper by Kaplan and Yorke [36] for estimating the Hausdorff dimension of attractors. The direct numerical computation of the Hausdorff dimension of attractors is often a problem of high numerical complexity (see, e.g. discussion in [73]), thus, various estimates of this dimension are of interest. Later the concept of the Lyapunov dimension has been developed in a number of papers (see, e.g. [14,24,26,28,32,35,45,77] and others).

Along with widely used numerical methods for estimating and computing the Lyapunov dimension there is an effective analytical approach, proposed by Leonov in 1991 [59] (see also [46,50,54,61,62]). The Leonov method is based on the direct Lyapunov method with special Lyapunov-like functions. The advantage of the Leonov method is that it allows one to estimate the Lyapunov dimension of invariant sets without localization of the set in the phase space and in many cases to get exact Lyapunov dimension formula [46,47,50,52,53,55,58,63].

Further the invariance of the Lyapunov dimension with respect to diffeomorphisms and its connection with the Leonov method are discussed. For discrete-time dynamical systems an analog of Leonov method is suggested.

2. Lyapunov dimension of maps and dynamical systems

Consider an autonomous differential equation

$$\dot{u} = f(u), \quad f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n, \tag{1}$$

where f is a continuously differentiable vector-function. Suppose that any solution $u(t,u_0)$ of (1) such that $u(0,u_0) = u_0 \in U$ exists for $t \in [0,\infty)$, it is unique and stays in U. Then the evolutionary operator $\varphi^t(u_0) = u(t,u_0)$ is continuously differentiable and satisfies the semigroup property:

$$\varphi^{t+s}(u_0) = \varphi^t(\varphi^s(u_0)),\ \varphi^0(u_0) = u_0 \quad \forall\, t,s \ge 0,\ \forall u_0 \in U. \tag{2}$$

Thus, $\{\varphi^t\}_{t\ge 0}$ is a smooth *dynamical system* in the phase space $(U, ||\cdot||)$: $\big(\{\varphi^t\}_{t\ge0}, (U \subseteq \mathbb{R}^n, ||\cdot||)\big)$. Here $||u|| = \sqrt{u_1^2 + \cdots + u_n^2}$ is Euclidean norm of the vector $u = (u_1,\ldots,u_n) \in \mathbb{R}^n$. Similarly, one can consider a dynamical system generated by the difference equation

$$u(t+1) = \varphi(u(t)), \quad t = 0,1,..,\tag{3}$$

where $\varphi : U \subseteq \mathbb{R}^n \to U$ is a continuously differentiable vector-function. Here $\varphi^t(u) = \underbrace{(\varphi \circ \varphi \circ \cdots \varphi)}_{t \text{ times}}(u)$, $\varphi^0(u) = u$, and the existence and uniqueness (in the forward-time direction) take place

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http://dx.doi.org/10.1016/j.physleta.2016.04.036
0375-9601/

for all $t \geq 0$. Further $\{\varphi^t\}_{t\geq 0}$ denotes a smooth *dynamical system* with continuous or discrete time.

Consider the linearizations of systems (1) and (3) along the solution $\varphi^t(u)$:

$$\dot{y} = J(\varphi^t(u))y, \quad J(u) = Df(u), \tag{4}$$

$$y(t+1) = J(\varphi^t(u))y(t), \quad J(u) = D\varphi(u), \tag{5}$$

where $J(u)$ is the $n \times n$ Jacobian matrix, the elements of which are continuous functions of u. Suppose that $\det J(u) \neq 0 \ \forall u \in U$.

Consider the fundamental matrix, which consists of linearly independent solutions $\{y^i(t)\}_{i=1}^n$ of the linearized system,

$$D\varphi^t(u) = \big(y^1(t), ..., y^n(t)\big), \quad D\varphi^0(u) = I, \tag{6}$$

where I is the unit $n \times n$ matrix. An important cocycle property of fundamental matrix (6) is as follows

$$D\varphi^{t+s}(u) = D\varphi^t\big(\varphi^s(u)\big)D\varphi^s(u), \ \forall t, s \geq 0, \ \forall u \in U. \tag{7}$$

Let $\sigma_i(t,u) = \sigma_i(D\varphi^t(u))$, $i = 1, 2, .., n$, be the singular values of $D\varphi^t(u)$ (i.e. $\sigma_i(t,u) > 0$ and $\sigma_i(t,u)^2$ are the eigenvalues of the symmetric matrix $D\varphi^t(u)^* D\varphi^t(u)$ with respect to their algebraic multiplicity), ordered so that $\sigma_1(t,u) \geq \cdots \geq \sigma_n(t,u) > 0$ for any $u \in U$, $t \geq 0$. The singular value function of order $d \in [0,n]$ at the point $u \in U$ for $D\varphi^t(u)$ is defined as

$$\omega_d(D\varphi^t(u)) = \begin{cases} 1, & d = 0, \\ \sigma_1(t,u)\sigma_2(t,u)\cdots\sigma_d(t,u), & d \in \{1,2,..,n\}, \\ \sigma_1(t,u)\cdots\sigma_{\lfloor d\rfloor}(t,u)\sigma_{\lfloor d\rfloor+1}(u)^{d-\lfloor d\rfloor}, & d \in (0,n), \end{cases} \tag{8}$$

where $\lfloor d\rfloor$ is the largest integer less or equal to d. Remark that $|\det D\varphi^t(u)| = \omega_n(D\varphi^t(u))$. Similarly, we can introduce the singular value function for arbitrary quadratic matrices. By the Horn inequality [31] for any two $n \times n$ matrices A and B and any $d \in [0,n]$ we have (see, e.g. [10, p. 28])

$$\omega_d(AC) \leq \omega_d(A)\omega_d(C), \quad d \in [0,n]. \tag{9}$$

Let a nonempty set $K \subset U \subseteq \mathbb{R}^n$ be invariant with respect to the dynamical system $\{\varphi^t\}_{t\geq 0}$, i.e. $\varphi^t(K) = K$ for all $t > 0$. Since in the numerical experiments only finite time t can be considered, for a fixed $t \geq 0$ let us consider the map defined by the evolutionary operator $\varphi^t(u)$: $\varphi^t : U \subseteq \mathbb{R}^n \to U$.

The concept of the Lyapunov dimension was suggested in the seminal paper by Kaplan and Yorke [36] and later it was developed in a number of papers (see, e.g. [14,23,26,32,45]). The following definition is inspirited by Douady and Oesterlé [20].

Definition 1. The local Lyapunov dimension[1] of the map φ^t (or finite-time local Lyapunov dimension of the dynamical system $\{\varphi^t\}_{t\geq 0}$) at the point $u \in U$ is defined as

$$\dim_{\mathrm{L}}(\varphi^t, u) = \inf\{d \in [0,n] : \omega_d(D\varphi^t(u)) < 1\}. \tag{10}$$

If the infimum is taken over an empty set (i.e. $\omega_n(D\varphi^t(u)) \geq 1$), we assume that the infimum and considered dimension are taken equal[2] to n.

The Lyapunov dimension of the map φ^t (or finite-time Lyapunov dimension of the dynamical system $\{\varphi^t\}_{t\geq 0}$) with respect to the invariant set K is defined as

$$\dim_{\mathrm{L}}(\varphi^t, K) = \sup_{u\in K} \dim_{\mathrm{L}}(\varphi^t, u) = \sup_{u\in K} \inf\{d \in [0,n] : \omega_d(D\varphi^t(u)) < 1\}. \tag{11}$$

The continuity of the functions $u \mapsto \sigma_i(D\varphi^t(u))$, $i = 1, 2, .., n$, on U implies that for any $d \in [0,n]$ and $t \geq 0$ the function $u \mapsto \omega_d(D\varphi^t(u))$ is continuous on U (see, e.g. [20], [27, p. 554]). Therefore for a compact set $K \subset U$ and $t \geq 0$ we have

$$\sup_{u\in K} \omega_d(D\varphi^t(u)) = \max_{u\in K} \omega_d(D\varphi^t(u)). \tag{12}$$

By relation (12) for a compact invariant set K one can prove that

$$\dim_{\mathrm{L}}(\varphi^t, K) = \inf\{d \in [0,n] : \max_{u\in K} \omega_d(D\varphi^t(u)) < 1\}. \tag{13}$$

In the seminal paper [20] Douady and Oesterlé proved rigorously that the Lyapunov dimension of the map φ^t with respect to the compact invariant set K is an upper estimate of the Hausdorff dimension of the set K:

$$\dim_{\mathrm{H}} K \leq \dim_{\mathrm{L}}(\varphi^t, K). \tag{14}$$

For numerical estimations of dimension, the following remark is important. From (7) and (9) it follows that

$$\sup_{u\in K} \omega_d(D\varphi^{t+s}(u)) = \sup_{u\in K} \omega_d\big(D\varphi^t(\varphi^s(u))D\varphi^s(u)\big) \leq \sup_{u\in K} \omega_d(D\varphi^t(u)) \sup_{u\in K} \omega_d(D\varphi^s(u)) \quad \forall t, s \geq 0$$

and $\sup_{u\in K} \omega_d(D\varphi^{nt}(u)) \leq (\sup_{u\in K} \omega_d(D\varphi^t(u)))^n$ for any integer $n \geq 0$. Thus for any $t \geq 0$ there exists $s(t) > 0$ such that

$$\dim_{\mathrm{L}}(\varphi^{t+s(t)}, K) \leq \dim_{\mathrm{L}}(\varphi^t, K). \tag{15}$$

Remark that if $\sup_{u\in K} \omega_d(D\varphi^t(u)) < 1$ for a certain $d \in [0,n]$, then

$$\inf_{t>0} \sup_{u\in K} \omega_d(D\varphi^t(u)) = \liminf_{t\to+\infty} \sup_{u\in K} \omega_d(D\varphi^t(u)) = 0. \tag{16}$$

While in the computations we can consider only finite time t and the map φ^t, from a theoretical point of view, it is interesting to study the limit behavior of finite-time Lyapunov dimension of the dynamical system $\{\varphi^t\}_{t\geq 0}$ with respect to the compact invariant set K.

Definition 2. The Lyapunov dimension of the dynamical system $\{\varphi^t\}_{t\geq 0}$ with respect to the invariant set K is defined as

$$\dim_{\mathrm{L}}(\{\varphi^t\}_{t\geq 0}, K) = \inf_{t>0} \dim_{\mathrm{L}}(\varphi^t, K). \tag{17}$$

From (14) and (13) we have

$$\dim_{\mathrm{H}} K \leq \dim_{\mathrm{L}}(\{\varphi^t\}_{t\geq 0}, K) = \inf_{t>0} \sup_{u\in K} \dim_{\mathrm{L}}(\varphi^t, u) \tag{18}$$

and (15) implies

$$\inf_{t>0} \dim_{\mathrm{L}}(\varphi^t, K) = \liminf_{t\to+\infty} \dim_{\mathrm{L}}(\varphi^t, K). \tag{19}$$

Definition 3. (See, e.g. [1].) The finite-time Lyapunov exponents (or the Lyapunov exponent functions of singular values) of the dynamical system $\{\varphi^t\}_{t\geq 0}$ at the point $u \in U$ are denoted by $\mathrm{LE}_i(t,u) = \mathrm{LE}_i(D\varphi^t(u))$, $i = 1, 2, .., n$, and defined as

$$\mathrm{LE}_i(t,u) = \frac{1}{t} \ln \sigma_i(t,u), \quad t > 0.$$

[1] This is not a dimension in a rigorous sense (see, e.g. [4,33,40]). The notion '*local Lyapunov dimension*' is used, e.g. in [22,32].

[2] In general, since $\omega_0(D\varphi^t(u)) \equiv 1$ and $d \mapsto \omega_d(D\varphi^t(u))$ is a left-continuous function, we have $\dim_{\mathrm{L}}(\varphi^t, u) = \max\{d \in [0,n] : \omega_d(D\varphi^t(u)) \geq 1\}$. If all $\{\sigma_i(t,u)\}_1^n$ are assumed to be positive and $\omega_n(D\varphi^t(u)) < 1$, then in (10) the infimum is achieved (see (20) and the Kaplan–Yorke formula (22)).