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Novel PT-invariant solutions for a large number of real nonlinear equations

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ABSTRACT

For a large number of real nonlinear equations, either continuous or discrete, integrable or nonintegrable, we show that whenever a real nonlinear equation admits a solution in terms of $\operatorname{sech} x$, it also admits solutions in terms of the PT-invariant combinations $\operatorname{sech} x \pm i \tanh x$. Further, for a number of real nonlinear equations we show that whenever a nonlinear equation admits a solution in terms $\operatorname{sech}^2 x$, it also admits solutions in terms of the PT-invariant combinations $\operatorname{sech}^2 x \pm i \operatorname{sech} x \tanh x$. Besides, we show that similar results are also true in the periodic case involving Jacobi elliptic functions.

1. Introduction

Nonlinear equations are playing an increasingly important role in several areas of science in general and physics in particular [1]. One of the major problems with these equations is the lack of a superposition principle. It may be thus desirable to explicitly obtain as many solutions of a given nonlinear equation as possible. It is, however, possible that auto-Bäcklund transformations may also provide such solutions in the case of integrable [2] and non-integrable [3] models. Thus, if we can find some general results about the existence of solutions to a nonlinear equation, that would be invaluable, at least in the case of non-integrable equations. In this context it is worth recalling that some time ago we [4,5] had shown (through a number of examples) that if a nonlinear equation admits a periodic solution in terms of Jacobi elliptic functions $\operatorname{dn}(x,m)$ and $\operatorname{cn}(x,m)$, then it will also admit solutions in terms of $\operatorname{dn}(x,m) \pm \sqrt{m}\operatorname{cn}(x,m)$, where m is the modulus of the elliptic function [6]. Further, in the same papers [4,5], we also showed (again through several examples) that if a nonlinear equation admits a solution in terms of $\operatorname{dn}^2(x,m)$, then it will also admit solutions in terms of $\operatorname{dn}^2(x,m) \pm \sqrt{m}\operatorname{cn}(x,m)\operatorname{dn}(x,m)$.

The purpose of this paper is to propose general results about the existence of new solutions to real nonlinear equations, integrable or nonintegrable, continuous or discrete through the idea of parity-time reversal or PT symmetry. It may be noted here that in the last 15 years or so the idea of PT symmetry [7] has given us new insights. In quantum mechanics it has been shown that even if a Hamiltonian is not hermitian but if it is PT-invariant, then the energy eigenvalues are still real in case the PT symmetry is not broken spontaneously. Further, there has been a tremendous growth in the number of studies of open systems which are specially balanced by PT symmetry [8–10] in several PT-invariant open systems bearing both loss and gain. In particular, many researchers have obtained soliton solutions which have been shown to be stable within a certain parameter range [11–13].

In this paper we highlight one more novel aspect of PT symmetry. Specifically, we obtain new PT-invariant solutions of several real nonlinear equations. Our strategy will be to start with known real solutions and then make Ansätze for complex PT-invariant solutions and obtain conditions under which the Ansätze are valid. We show, through several examples, that whenever a real nonlinear equation, either continuous or discrete, integrable or nonintegrable, admits a solution in terms of $\operatorname{sech} x$, then it will necessarily also admit solutions in terms of the PT-invariant combinations $\operatorname{sech} x \pm i \tanh x$. We also generalize these results to the periodic case and show that whenever a nonlinear equation admits a solution in terms of $\operatorname{dn}(x,m)$ [or $\operatorname{cn}(x,m)$], then it will necessarily also admit solutions in terms of the PT-invariant combinations $\operatorname{dn}(x,m) \pm i\sqrt{m}\operatorname{sn}(x,m)$ [or $\operatorname{cn}(x,m) \pm i \operatorname{sn}(x,m)$].

In addition, we also show through several examples that whenever a real nonlinear equation admits a solution in terms of $\operatorname{sech}^2 x$, then it will also admit solutions in terms of $\operatorname{sech}^2 x \pm i \operatorname{sech} x \tanh x$. We also generalize these results to the periodic case and show that whenever a real nonlinear equation admits a solution in terms of $\operatorname{dn}^2(x,m)$, then it will necessarily also admit

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solutions in terms of $\mathrm{dn}^2(x,m) \pm im\mathrm{sn}(x,m)\,\mathrm{cn}(x,m)$ as well as $\mathrm{dn}^2(x,m) \pm i\sqrt{m}\,\mathrm{sn}(x,m)\,\mathrm{dn}(x,m)$. These results may find physical realization in coupled optical waveguides among other applications [14]. It is worth pointing out that some of the equations considered here have also been considered in their PT-symmetric deformed version [15].

2. Solutions in terms of $\mathrm{sech}\,x \pm i\tanh x$ and their periodic generalization

We now discuss four examples, two from continuum field theories and two from the discrete case where $\mathrm{sech}\,x$ is a known solution and in all the four cases we obtain new PT-invariant solutions in terms of $\mathrm{sech}\,x \pm i\tanh x$ and also periodic PT-invariant solutions in terms of $\mathrm{dn}(x,m) \pm i\,\mathrm{sn}(x,m)$ as well as $\mathrm{cn}(x,m) \pm i\,\mathrm{sn}(x,m)$.

2.1. ϕ^4 field theory

The ϕ^4 field theory arises in several areas of physics [16] including second order phase transitions. The field equation for the ϕ^2–ϕ^4 field theory is given by

$$\phi_{xx} = a\phi + b\phi^3\,. \tag{1}$$

In case $b < 0$, one of the well known solutions to this equation is

$$\phi = A\,\mathrm{sech}[\beta x]\,, \tag{2}$$

provided

$$bA^2 = -2\beta^2\,,\ \ a = \beta^2\,. \tag{3}$$

Remarkably, even

$$\phi = A\,\mathrm{sech}(\beta x) \pm iB\tanh(\beta x) \tag{4}$$

is an exact PT-invariant solution of Eq. (1) provided

$$B = \pm A\,,\ \ 2bA^2 = -\beta^2\,,\ \ a = -(1/2)\beta^2\,. \tag{5}$$

Further, as we now show, such PT-invariant solutions also exist in the periodic case. Let us first note that one of the exact, periodic solutions to the ϕ^4 Eq. (1) is [17]

$$\phi = A\,\mathrm{dn}(\beta x, m)\,, \tag{6}$$

provided

$$bA^2 = -2\beta^2\,,\ \ a = (2-m)\beta^2\,. \tag{7}$$

Further, the same model (1) is known to admit another periodic solution

$$\phi = A\sqrt{m}\,\mathrm{cn}(\beta x, m)\,, \tag{8}$$

provided

$$bA^2 = -2\beta^2\,,\ \ a = (2m-1)\beta^2\,. \tag{9}$$

Remarkably, we find that the same model also admits the PT-invariant periodic solution

$$\phi = A\,\mathrm{dn}(\beta x, m) + iB\sqrt{m}\,\mathrm{sn}(\beta x, m)\,, \tag{10}$$

provided

$$B = \pm A\,,\ \ 2bA^2 = -\beta^2\,,\ \ a = -\frac{2m-1}{2}\beta^2. \tag{11}$$

Further, the same model also admits another PT-invariant solution

$$\phi = A\sqrt{m}\,\mathrm{cn}[\beta x, m] + iB\sqrt{m}\mathrm{sn}[\beta x, m]\,, \tag{12}$$

provided

$$B = \pm A\,,\ \ 2bA^2 = -\beta^2\,,\ \ a = -\frac{2-m}{2}\beta^2\,. \tag{13}$$

2.2. mKdV equation

We now discuss the celebrated modified Korteweg–de Vries (mKdV) equation

$$u_t + u_{xxx} + 6u^2u_x = 0\,, \tag{14}$$

which is a well known integrable equation having application in several areas [16]. It is well known [16] that

$$u = A\,\mathrm{sech}[\beta(x - vt)] \tag{15}$$

is an exact solution of Eq. (14) provided

$$A^2 = \beta^2\,,\ v = \beta^2\,. \tag{16}$$

Remarkably, even

$$u = A\,\mathrm{sech}[\beta(x - vt)] \pm iB\tanh[\beta(x - vt)] \tag{17}$$

is also an exact PT-invariant solution to the mKdV Eq. (14) provided

$$B = \pm A\,,\ \ A^2 = 4\beta^2\,,\ \ v = -(1/2)\beta^2\,. \tag{18}$$

Even more remarkable, such PT-invariant solutions also exist in the periodic case. For example, it is well known that one of the exact, periodic solutions to the mKdV Eq. (14) is [16]

$$u = A\,\mathrm{dn}[\beta(x - vt), m]\,, \tag{19}$$

provided

$$A^2 = \beta^2\,,\ \ v = (2-m)\beta^2\,. \tag{20}$$

Another periodic solution to the mKdV Eq. (14) is

$$u = A\sqrt{m}\,\mathrm{cn}[\beta(x - vt), m]\,, \tag{21}$$

provided

$$A^2 = \beta^2\,,\ \ v = (2m-1)\beta^2\,. \tag{22}$$

Remarkably, even

$$u = A\,\mathrm{dn}[\beta(x - vt), m] + iB\sqrt{m}\,\mathrm{sn}[\beta(x - vt), m] \tag{23}$$

is an exact PT-invariant solution to the mKdV Eq. (14) provided

$$B = \pm A\,,\ \ A^2 = 4\beta^2\,,\ \ v = -\frac{(2m-1)}{2}\beta^2\,. \tag{24}$$

We thus have two new periodic solutions of mKdV Eq. (14) depending on whether $B = A$ or $B = -A$.

Further, even

$$u = A\sqrt{m}\,\mathrm{cn}[\beta(x - vt), m] + iB\sqrt{m}\mathrm{sn}[\beta(x - vt), m]\,, \tag{25}$$

is an exact PT-invariant solution of the mKdV Eq. (14) provided

$$B = \pm A\,,\ \ A^2 = 4\beta^2\,,\ \ v = -\frac{(2-m)}{2}\beta^2\,. \tag{26}$$

2.3. Discrete ϕ^4 equation

We now discuss two discrete models and show that both these models also admit PT-invariant solutions. Let us first consider the discrete ϕ^4 equation

$$\frac{1}{h^2}[\phi_{n+1} + \phi_{n-1} - 2\phi_n] + a\phi_n - \frac{\lambda}{2}\phi_n^2[\phi_{n+1} + \phi_{n-1}] = 0\,. \tag{27}$$

It is well known that Eq. (27) admits an exact solution [18]

$$\phi_n = A\,\mathrm{sech}(\beta n)\,, \tag{28}$$