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Steady and perturbed motion of a point vortex along a boundary with a circular cavity

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ABSTRACT

The dynamics of a point vortex moving along a straight boundary with a circular cavity subjected to a background flow is investigated. Given the constant background flow, this configuration produces regular phase portraits of the vortex motion. These phase portraits are discriminated depending on the cavity's circular shape, and then the transition to chaos of the vortex motion is investigated given an oscillating perturbation superimposed on the background flow. Based on the steady-state vortex rotation, the forcing parameters that lead to effective destabilization of vortex trajectories are distinguished. We show that, provided the cavity aperture is relatively narrow, the periodic forcing superimposed on the background flow destabilizes the vortex trajectories very slightly. On the other hand, if the cavity aperture is relatively wide, the forcing can significantly destabilize vortex trajectories causing the majority of the trajectories, which would be closed without the forcing, to move towards infinity.

1. Introduction

The motion of point vortices in domains of curved geometry is of interest partially because of possible applications. In the ocean one can rarely observe a straight boundary. Most of the boundaries are curved, which thus significantly alters the vortex dynamics in their immediate vicinity, sometimes forcing eddies to stay inside bays [1–3]. For instance, very intense mesoscale dynamics is often observed in the Bay of Biscay. Making use of satellite images, the authors of [4] reconstructed trajectories of isolated vortices moving along the continental slope. Some of these vortices remain coherent for very long, at least, until they left the bay. The authors of [2] modeled numerically the interaction of surface and deep vortices in the same area, and established the features of the vortex interaction depending on the distance to the slope. Experimental measurements [5] indicated the presence of coherent vortex structures near the Gulf of Lions, which are entrapped inside the gulf. Numerical simulation of the circulation and radar observation in the Gulf of Lions [1] also revealed persistent mesoscale vortex structures that move along the curved coastline and are able to pass over the gulf without disintegrating. These examples illustrate that coherent vortex structures can persist for a long time, only slightly changing shape, near curved boundaries. This observation is important because it allows us to expect that using the simplest vortex model, the point vortex model in a potential fluid, may provide valuable qualitative insight about feasible vortex motion near curved coastlines.

Moreover, the dynamics of point vortices in domains with curved boundaries is interesting in itself. The boundaries change the point-vortex trajectory topology, which may result in the appearance of periodic motion [6–11]. A point vortex moves in a rectilinear, uniform fashion along a straight wall because this configuration is equivalent to the motion of a vortex pair consisting of equal co- and counter rotating point vortices [12]. To change the geometry of the boundary, one then needs to map conformally the straight boundary into the desired geometry. However, to ensure that the point vortex in the new geometry preserves its strength, one should make use the Kirchhoff–Routh stream-function [13,14].

This configuration is chosen to mimic the simplest shape of a generic bay. Using this idealized model, it is possible to gain insights into the ways an oceanic bay can entrap vortex structures.

2. Problem formulation

We consider the following conformal mapping, that maps a straight line with a circular cavity of radius R into the upper half-plane,

$$\zeta = b\frac{1}{1-\gamma z_2}, \quad z_2 = z_1^{\frac{1}{2-\alpha}}, \quad z_1 = \frac{z-a}{z+a}, \qquad (1)$$

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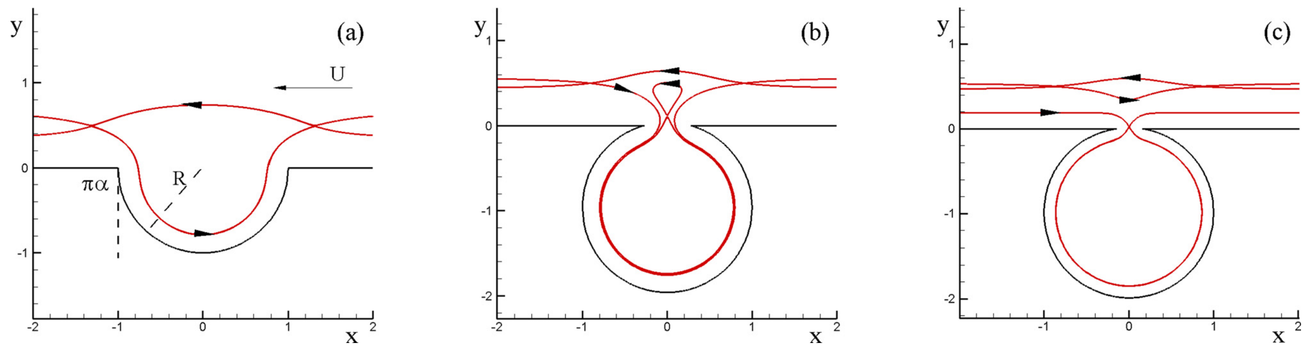

Fig. 1. Separatrices of the steady-state vortex motion, which forms typical phase portraits as $U_0 = -1$, $\epsilon = 0$, $\mu = 1$: (a) $\alpha = 0.5$, (b) $\alpha = 0.09$, (c) $\alpha = 0.05$.

where $a = R\sin(\pi\alpha)$, $b = 2a/(2-\alpha)$, $\pi\alpha$ is the characteristic angle of the cavity (see Fig. 1a), $z = x + iy$ is the complex variable in the original domain with the cavity, and $\zeta = \xi + i\eta$ is the complex variable in the mapped upper-plane domain. Here $\gamma = 1$ if $y \geq 0$ and $\gamma = \exp\{2i\pi/(2-\alpha)\}$ if $y < 0$ defining the corresponding branch of the mapping.

A point vortex with a strength μ then moves in the z-plane according to the Kirchhoff–Routh stream-function [13,14], which yields the following complex velocity field for the vortex motion,

$$\frac{dz_v^*}{dt} = \left.\frac{d\zeta}{dz}\right|_{z=z_v}\left(U + \frac{\mu}{2\eta_v(z_v)}\right) - i\mu\frac{\left.\frac{d^2\zeta}{dz^2}\right|_{z=z_v}}{2\left.\frac{d\zeta}{dz}\right|_{z=z_v}}, \tag{2}$$

where the subscript v marks the coordinates of the vortex, and $\cdot^*$ is the complex conjugate. The term $U + \frac{\mu}{2\eta_v}$ corresponds to the influence of the mirror vortex [12] and a plane flow with velocity U, while the term $i\mu\frac{\left.\frac{d^2\zeta}{dz^2}\right|_{z=z_v}}{2\left.\frac{d\zeta}{dz}\right|_{z=z_v}}$ appears because the vortex strength in the domain z should be equal to the vortex strength in the domain ζ. Then, by solving the governing equations

$$\frac{dx_v}{dt} = \mathrm{Re}\,\frac{dz_v^*}{dt}, \quad \frac{dy_v}{dt} = -\,\mathrm{Im}\,\frac{dz_v^*}{dt}, \tag{3}$$

one obtains the vortex trajectories.

The complex velocity of a fluid particle in the z-plane is

$$\frac{dz^*}{dt} = \frac{d\zeta}{dz}\left(U + i\mu\left(\frac{1}{\zeta - \zeta_v} - \frac{1}{\zeta - \zeta_v^*}\right)\right). \tag{4}$$

The fluid particle trajectories can then be calculated using the following equations, analogous to (3),

$$\frac{dx}{dt} = \mathrm{Re}\,\frac{dz^*}{dt}, \quad \frac{dy}{dt} = -\,\mathrm{Im}\,\frac{dz^*}{dt}. \tag{5}$$

3. The steady-state dynamics of the point vortex

Near a straight wall, a vortex induces a bubble that does not mix with outer fluid, and carries it towards infinity owing to the self-induced rectilinear translation of the vortex. If one superimposes a plane background flow directed along the wall, the vortex will just move faster or slower depending on the background flow direction. Nevertheless, its motion will still be rectilinear and uniform. Change in the dynamics occurs if the background flow is sheared [15]. Then, the vortex starts oscillating periodically along its path, which leads to stirring of the fluid in its immediate vicinity. Nevertheless, in both cases the vortex translates monotonically to infinity if its average self-propulsion velocity is not equal to the average background flow velocity.

Now, by setting a curved wall, one forces the vortex to change its path [11]. Depending on the curvature, there can appear a new type of dynamics, such that the vortex can start circulating periodically in certain localized regions. In other words, the vortex can be entrapped inside the cavity or can pass it continuing moving with the flow. The first regime is especially interesting since periodic oscillations of the vortex can be perturbed giving rise to chaotic dynamics. In this case, the perturbation is superimposed as a periodically changing external flow. Let the background flow be of the form

$$U = U_0(1 + \varepsilon\sin\nu t), \tag{6}$$

where U_0 is the mean flow, ε, ν are the forcing magnitude and frequency.

Because the system has many parameters, we restrict ourselves to examine the dynamics depending on the cavity's characteristic angle α, whereas the strength $\mu = 1$ and the background flow $U_0 = -1$ throughout the paper.

To start, let us look into the steady-state system as $\varepsilon = 0$. Hence, the background flow is assumed to be constant $U = U_0$. There can be discriminated three types of the phase space structure that differ by the number of critical points. The first one comprises three critical points, one is elliptic and the other two are hyperbolic (see Fig. 1a). The second one has two elliptic and three hyperbolic critical points (see Fig. 1b). The third one also features the same set of critical points with two elliptic and three hyperbolic ones, but the structure of the phase portrait is different. The discrepancy is that there appear two distinctively separate closed recirculation zones shown in Fig. 1c instead of one closed region with an inner separatrix nested inside an outer separatrix (see Fig. 1b).

It is feasible to demonstrate that the classification of the phase space portrait relying on the number of the critical points is complete. To determine the position of the critical points, one needs to equate eq. (2) to zero and then to solve the resulting equation. However, for convenience, we take advantage of the inverse mapping $z_v = f(\zeta_v)$, which yields the governing equation in the ζ-plane,

$$\frac{d\zeta_v*}{dt} = \frac{1}{|f'(\zeta_v)|^2}\left[U + \mu i\frac{1}{\zeta_v - \zeta_v*} + \frac{1}{2}\mu i\frac{f''(\zeta_v)}{f'(\zeta_v)}\right], \tag{7}$$

where the inverse mapping implicitly reads

$$z_v = f(\zeta_v) = -a\left[\left(1 - \frac{b}{\zeta_v}\right)^{2-\alpha} + 1\right] \Big/ \left[\left(1 - \frac{b}{\zeta_v}\right)^{2-\alpha} - 1\right]. \tag{8}$$

Thus,

$$\frac{U}{\mu} + \frac{1}{2\eta_v} - \frac{1}{2}i\frac{f''(\zeta_v)}{f'(\zeta_v)} = 0. \tag{9}$$