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From the hypergeometric differential equation to a non-linear Schrödinger one

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ABSTRACT

We show that the q-exponential function is a hypergeometric function. Accordingly, it obeys the hypergeometric differential equation. We demonstrate that this differential equation can be transformed into a non-linear Schrödinger equation (NLSE). This NLSE exhibits both similarities and differences vis-a-vis the Nobre–Rego-Monteiro–Tsallis one.

1. Introduction

In 2011, Nobre, Rego-Monteiro and Tsallis (NRT) [1–5] introduced an intriguing new version of the nonlinear Schrödinger equation (NLSE), an interesting proposal that one may regard as part of a project to explore non-linear versions of some of the fundamental equations of physics, a research venue actively visited in recent times [6,7]. Earlier non-linear versions of the SE have found application in diverse areas (fiber optics and water waves, for instance) [7]. A most studied NLSE involves a cubic nonlinearity in the wave function. In quantum settings the NLSE usually rules the behavior of a single-particle's wave function that, in turn, provides an effective, mean-field description of a quantum many-body system. An important case is the Gross–Pitaevskii equation, employed in researching Bose–Einstein condensates [8]. The cubic nonlinear term appearing in the Gross–Pitaevskii equation describes short-range interactions between the condensate's constituents. The NLSE for the system's (effective) single-particle wave function is found assuming a Hartree–Fock-like form for the global many-body wave function, with a Dirac's delta form for the inter-particle potential.

The NRT equation derives from the thermo-statistical formalism based upon the Tsallis S_q non-additive, power-law information measure. Applications of the functional S_q involve diverse physical systems and processes, having attracted much attention in the last 20 years (see, for example, [9–17], and references therein). In particular, the S_q entropy has proved to be useful for the analysis of diverse problems in quantum physics [18–26].

In this paper we traverse a totally different road. We start from the differential equation that governs hypergeometric functions and derive from it a new NLSE that is different from, but exhibits some similarities with, the NRT.

2. A new non-linear Schrödinger equation

The q-exponential e_q is defined as $e_q(x) = [1+(q-1)x]_+^{\frac{1}{1-q}}$, that is,

$$\begin{aligned} e_q(x) &= [1+(q-1)x]_+^{\frac{1}{1-q}} \\ &= [1+(q-1)x]^{\frac{1}{1-q}} \text{ if } 1+(q-1)x>0 \\ e_q(x) &= 0, \quad \text{otherwise} \quad (\text{with } q\in\mathcal{R}). \end{aligned} \tag{2.1}$$

A search in [27] reveals that

$$F(-\alpha,\gamma;\gamma;-z) = (1+z)^\alpha, \tag{2.2}$$

which yields for $e_q[(i/\hbar)(px-Et)] \equiv e_q(Y)$ the relation (with $E=\frac{p^2}{2m}$)

$$\begin{aligned} &\left[1+\frac{i}{\hbar}(1-q)(px-Et)\right]^{\frac{1}{1-q}} \\ &\quad = F\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right], \end{aligned} \tag{2.3}$$

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which is a fundamental result for us. We consider below derivatives F' and F'' of F with respect to Y.

Now, according to [28], the hypergeometric function obeys the following, differential equation (primes denote derivatives with respect to Y)

$$z(1-z)F''(\alpha,\beta;\gamma;z)+ \\ +[\gamma-(\alpha+\beta+1)z]F'(\alpha,\beta;\gamma;z)-\alpha\beta F(\alpha,\beta;\gamma;z)=0, \tag{2.4}$$

so that, specializing things for our instance (2.3) we encounter

$$\begin{aligned}&\frac{i}{\hbar}(q-1)(px-Et)\left[1-\frac{i}{\hbar}(q-1)(px-Et)\right]\\&\quad\times F''\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right]\\&\quad+\left[\gamma-\left(\frac{1}{q-1}+\gamma+1\right)\frac{i}{\hbar}(q-1)(px-Et)\right]\\&\quad\times F'\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right]\\&\quad-\frac{\gamma}{q-1}F\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right]=0.\end{aligned} \tag{2.5}$$

This allows for a relation between the derivative with respect to the argument and the partial derivative with respect to time, for this hypergeometric function

$$\begin{aligned}&F'\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right]\\&\quad=\frac{i\hbar}{(q-1)E}\frac{\partial}{\partial t}F\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right].\end{aligned} \tag{2.6}$$

In analogous fashion we obtain, for the second partial derivative with respect to the position

$$\begin{aligned}&F''\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right]\\&\quad=-\frac{\hbar^2}{(q-1)^2p^2}\frac{\partial^2}{\partial x^2}F\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right].\end{aligned} \tag{2.7}$$

Replacing (2.6) and (2.7) into (2.5), this last equation adopts the appearance

$$\begin{aligned}&-\frac{i}{\hbar}(q-1)(px-Et)\left[1-\frac{i}{\hbar}(q-1)(px-Et)\right]\\&\quad\times\frac{\hbar^2}{(q-1)^2p^2}\frac{\partial^2}{\partial x^2}F\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right]\\&\quad+\left[\gamma-\left(\frac{1}{q-1}+\gamma+1\right)\frac{i}{\hbar}(q-1)(px-Et)\right]\\&\quad\times\frac{i\hbar}{(q-1)E}\frac{\partial}{\partial t}F\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right]\\&\quad-\frac{\gamma}{q-1}F\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right]=0,\end{aligned} \tag{2.8}$$

that can be recast in the fashion

$$\begin{aligned}&-\frac{i}{\hbar}(q-1)(px-Et)\left[1-\frac{i}{\hbar}(q-1)(px-Et)\right]\\&\quad\times\frac{\hbar^2}{(q-1)m^2}\frac{\partial^2}{\partial x^2}F\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right]\\&\quad+\left[\gamma-\left(\frac{1}{q-1}+\gamma+1\right)\frac{i}{\hbar}(q-1)(px-Et)\right]\\&\quad\times i\hbar\frac{\partial}{\partial t}F\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right]\\&\quad-\gamma EF\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right]=0.\end{aligned} \tag{2.9}$$

Deriving (2.3) with respect to time we obtain:

$$\begin{aligned}&-\gamma EF\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right]\\&\quad=-i\hbar\gamma\left\{F\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right]\right\}^{(1-q)}\\&\quad\times\frac{\partial}{\partial t}F\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right].\end{aligned} \tag{2.10}$$

For simplicity, let us abbreviate

$$F\equiv F\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right]. \tag{2.11}$$

Using now (2.10), Eq. (2.9) becomes

$$\begin{aligned}&-\frac{\hbar^2}{2m(q-1)}\left[1-F^{(1-q)}\right]F^{(1-q)}\frac{\partial^2}{\partial x^2}F\\&\quad+i\hbar\left\{\gamma+\left(\frac{1}{q-1}+\gamma+1\right)\left[F^{(1-q)}-1\right]\right\}\frac{\partial}{\partial t}F\\&\quad-i\hbar\gamma F^{(1-q)}\frac{\partial}{\partial t}F=0.\end{aligned} \tag{2.12}$$

Simplifying things in this last relation we arrive at

$$-\frac{\hbar^2}{2m}F^{(1-q)}\frac{\partial^2}{\partial x^2}F-i\hbar q\frac{\partial}{\partial t}F=0, \tag{2.13}$$

that can be rewritten as

$$i\hbar q\frac{\partial}{\partial t}F=F^{(1-q)}H_0F, \tag{2.14}$$

where H_0 is the free particle Hamiltonian, note that, for $q=1$, one reobtains Schrödinger's free particle equation. Now, if instead of (2.3) we deal just with

$$F(x,t)=A\left[1+\frac{i}{\hbar}(1-q)(px-Et)\right]^{\frac{1}{1-q}}, \tag{2.15}$$

then $F(0,0)=A$ and (2.14) becomes

$$i\hbar q\frac{\partial}{\partial t}\left[\frac{F(x,t)}{F(0,0)}\right]=\left[\frac{F(x,t)}{F(0,0)}\right]^{(1-q)}H_0\left[\frac{F(x,t)}{F(0,0)}\right], \tag{2.16}$$

or, equivalently,

$$i\hbar q\left[\frac{F(x,t)}{F(0,0)}\right]^{(q-1)}\frac{\partial}{\partial t}\left[\frac{F(x,t)}{F(0,0)}\right]=H_0\left[\frac{F(x,t)}{F(0,0)}\right], \tag{2.17}$$

that, in turn can be recast as

$$i\hbar\frac{\partial}{\partial t}\left[\frac{F(x,t)}{F(0,0)}\right]^q=H_0\left[\frac{F(x,t)}{F(0,0)}\right]. \tag{2.18}$$

At this stage we realize that this last equation could be 'generalized' to any Hamiltonian H as

$$i\hbar\frac{\partial}{\partial t}\left[\frac{\psi(x,t)}{\psi(0,0)}\right]^q=H\left[\frac{\psi(x,t)}{\psi(0,0)}\right]. \tag{2.19}$$

With the change of variables $[\psi(x,t)]^q=\phi(x,t)$, Eq. (2.19) takes the form

$$i\hbar\frac{\partial}{\partial t}\left[\frac{\phi(x,t)}{\phi(0,0)}\right]=H\left[\frac{\phi(x,t)}{\phi(0,0)}\right]^{\frac{1}{q}}, \tag{2.20}$$

which trivially reduces to the ordinary Schrödinger equation for $q=1$.